Lifting systems of Galois representations associated to abelian varieties

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Abstract

This paper treats what we call 'weak geometric liftings' of Galois representations associated to abelian varieties. This notion can be seen as a generalization of the idea of lifting a Galois representation along an isogeny of algebraic groups. The weaker notion only takes into account an isogeny of the derived groups and disregards the centres of the groups in question. The weakly lifted representations are required to be geometric in the sense of a conjecture of Fontaine and Mazur. The conjecture in question states that any irreducible geometric representation is a twist of a subquotient of an étale cohomology group of an algebraic variety over a number field.

It is shown that a Galois representation associated to an abelian variety admits a weak geometric lift to a group with simply connected derived group. In certain cases, such a weak geometric lift is itself associated to an abelian variety. This means that the conjecture of Fontaine and Mazur is confirmed for these representations. In other cases, one may find a lift which can not be found back in the étale cohomology of any abelian variety. The Fontaine–Mazur conjecture remains open for these representations. Nevertheless, certain consequences of the conjecture can be established.

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Introduction

This paper is motivated by a conjecture of Fontaine and Mazur, conjecture 1 of [FM95] and generalizes the observations made in [Noo01]. The conjecture

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in question aims to characterize the Tate twists of the irreducible subquotients of the Galois representations arising from the étale cohomology of algebraic varieties over number fields.

To explain the conjecture, fix a number field F and let $\mathscr{G}_F$ be the absolute Galois group of F. In [FM95, §1], a representation of $\mathscr{G}_F$ on a finite dimensional $\mathbf{Q}_p$-vector space V_p is called *geometric* if it is unramified outside a finite set of non-archimedean places of F and if, for each non-archimedean valuation $\bar{v}$ of $\bar{F}$ of residue characteristic p, the restriction to the inertia group at $\bar{v}$ is potentially semi-stable in the sense of Fontaine, see also 4.1. An irreducible representation of $\mathscr{G}_F$ on V_p is said to *come from algebraic geometry* if it is isomorphic to a subquotient of the Galois representation on a cohomology group $\mathrm{H}^i_{\text{ét}}(X_{\bar{F}}, \mathbf{Q}_p)(m)$ for some smooth and proper F-scheme X and integers i and m. The Tate twist (m) has the effect of multiplying the action of $\mathscr{G}_F$ on $\mathrm{H}^i_{\text{ét}}(X_{\bar{F}}, \mathbf{Q}_p)$ by the mth power of the cyclotomic character. Fontaine and Mazur have expressed the conjecture that an irreducible $\mathbf{Q}_p$-linear representation V_p of $\mathscr{G}_F$ is geometric if and only if it comes from algebraic geometry.

The 'if'-part of the conjecture being resolved (see [Tsu99]), this paper is concerned with some reflexions on the implication 'only if' in the conjecture. We will actually investigate a particular type of geometric representations.

The construction of these representations starts with the choice of an abelian variety A over F. For such an abelian variety, the representation of $\mathscr{G}_F$ on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ factors through a map

$$\rho_{A,p} \colon \mathscr{G}_F \to G_A(\mathbf{Q}_p),$$

where G_A is the Mumford–Tate group of A. After reading [Win95], one is tempted to look for an isogeny $\pi \colon \widetilde{G} \to G_A$ and a map $\tilde{\rho}_p \colon \mathscr{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ such that $\rho_{A,p} = \pi \circ \tilde{\rho}_p$ and such that $\tilde{\rho}_p$ defines a geometric representation of $\mathscr{G}_F$ on $\widetilde{V}_p$ for any $\mathbf{Q}_p$-linear representation $\widetilde{V}_p$ of $\widetilde{G}_{/\mathbf{Q}_p}$.

This is not exactly the point of view adopted in this paper. As the conjecture of Fontaine–Mazur for representations with abelian image is quite well understood, cf. [FM95, §6], the centres of the groups occurring above can be considered to be less interesting than the derived groups. For this reason, given the representation $\rho_{A,p} \colon \mathscr{G}_F \to G_A(\mathbf{Q}_p)$, we will search for a linear algebraic group $\widetilde{G}$, an isogeny $\pi^{\mathrm{der}} \colon \widetilde{G}^{\mathrm{der}} \to G_A^{\mathrm{der}}$ and a map $\tilde{\rho}_p \colon \mathscr{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ such that $\pi^{\mathrm{ad}} \circ \rho_{A,p} = \tilde{\pi}^{\mathrm{ad}} \circ \tilde{\rho}_p$. Here $\pi^{\mathrm{ad}} \colon G_A \to G_A^{\mathrm{ad}}$ and $\tilde{\pi}^{\mathrm{ad}} \colon \widetilde{G} \to \widetilde{G}^{\mathrm{ad}}$ are the natural

projections and the groups G_A^{ad} and $\widetilde{G}^{\mathrm{ad}}$ have been identified using the map induced by π^{der}. If such a representation $\tilde{\rho}_p$ defines a geometric representation of $\mathscr{G}_F$ on any $\mathbf{Q}_p$-linear representation $\widetilde{V}_p$ of $\widetilde{G}_{/\mathbf{Q}_p}$, then we will say that it is a *weak geometric lift* of $\rho_{A,p}$. An advantage of working with weak geometric liftings is that, by 'correcting' liftings using characters, it is easier to produce geometric representations than if one considers liftings along isogenies.

It turns out that for any abelian variety A/F, there exist a finite extension F'/F, a group $\widetilde{G}$ such that $\widetilde{G}^{\mathrm{der}}$ is the universal cover of G_A and, for every prime number p, a weak geometric lift $\tilde{\rho}_p \colon \mathscr{G}_{F'} \to \widetilde{G}(\mathbf{Q}_p)$ of the restriction of $\rho_{A,p}$ to $\mathscr{G}_{F'}$. We refer to corollary 5.11 for a precise statement. Moreover, it follows from corollary 5.12 that every weak geometric lift of a $\rho_{A,p}$ is dominated by a weak geometric lift to a group $\widetilde{G}$ with $\widetilde{G}^{\mathrm{der}}$ simply connected. We can even conveniently 'normalize' the group $\widetilde{G}$.

It is natural to ask if the conjecture of Fontaine and Mazur is true for any weak geometric lift of a representation associated to an abelian variety. This question is not answered in this paper, but a number of partial results are obtained.

In section 4 we prove the following results. By combining proposition 3.4 and remark 4.10, it follows that for every abelian variety A/F, there exists a group $\widetilde{G}$, a map $\widetilde{G}^{\mathrm{der}} \to G_A^{\mathrm{der}}$ and, after replacing F by a finite extension, a system $(\tilde{\rho}_p)$ of weak geometric liftings of the $\rho_{A,p}$ such that

- the group $\widetilde{G}^{\mathrm{der}}$ is 'not far' from the universal cover of G_A^{der} and
- for any representation $\widetilde{V}$ of $\widetilde{G}$, the system of representations of $\mathscr{G}_F$ on the $\widetilde{V} \otimes \mathbf{Q}_p$ is isomorphic to the system of p-adic representation associated to an abelian variety.

To be more precise where the first property is concerned, it means in the first place that the group $\widetilde{G}^{\mathrm{der}}_{/\mathbf{C}}$ is the product of its simple factors $\widetilde{G}_i$. Secondly, it follows from well-known facts on the Mumford–Tate groups of abelian varieties that these factors are all of classical type (A, B, C or D). For the $\widetilde{G}_i$ which are of type A_k, B_k or C_k, being 'not far' from the universal cover means that they are simply connected. Where the factors $\widetilde{G}_i$ of type D_k are concerned, the condition is more difficult to state. We have to distinguish two subtypes, $D_k^{\mathbf{R}}$ and $D_k^{\mathbf{H}}$, see 2.5 for the definitions. The $\widetilde{G}_i$ which are of type $D_k^{\mathbf{R}}$ in this classification are also simply connected. The factors $\widetilde{G}_i$ of type $D_k^{\mathbf{H}}$ are *h-maximal* in the sense

of 2.5, which means that every such $\widetilde{G}_i$ is a quotient of its universal cover by a subgroup of order 2. Over $\mathbf{R}$, these h-maximal groups are orthogonal groups.

By theorem 4.9, the above system $(\tilde{\rho}_p)$ is maximal in the following sense. For any group G', any isogeny $G'^{\mathrm{der}} \to G_A^{\mathrm{der}}$ such that G'^{der} is a quotient of $\widetilde{G}^{\mathrm{der}}$ and any weak geometric lift $\rho'_p \colon \mathscr{G}_F \to G'(\mathbf{Q}_p)$ of $\rho_{A,p}$ (allowing a finite extension of F), the representation ρ'_p belongs to the tannakian category generated by the p-adic Galois representations associated to abelian varieties and the representations with finite image.

The above statements about Galois representations follow from analog properties of Hodge structures associated to abelian varieties. If $A_{/\mathbf{C}}$ is an abelian variety over $\mathbf{C}$, then the Hodge structure on the first Betti cohomology group $\mathrm{H}^1_{\mathrm{B}}(A_{/\mathbf{C}}(\mathbf{C}), \mathbf{Q})$ is determined by a morphism $h \colon S \to G_{A/\mathbf{R}}$, where $S = \mathbf{C}^\times$ as algebraic groups over $\mathbf{R}$. It is show in section 2 that the group $\widetilde{G}$ above is actually the Mumford–Tate group of an abelian variety $B_{/\mathbf{C}}$ and that the Hodge structure of B corresponds to a morphism $\tilde{h} \colon S \to \widetilde{G}_{/\mathbf{R}}$ such that h and $\tilde{h}$ have the same projection to $G^{\mathrm{ad}}_{A/\mathbf{R}} = \widetilde{G}^{\mathrm{ad}}_{/\mathbf{R}}$. To underscore the analogy with the construction of weak geometric liftings of Galois representations, such an abelian variety B will be called a *weak Mumford–Tate lift* of A, cf. 2.1. This notion is very close to the notion of $A_{/\mathbf{C}}$ and $B_{/\mathbf{C}}$ being 'adjoint-isogenous' in the sense of definition 6.1 of Vasiu's e-print [Vas03] and our variety $B_{/\mathbf{C}}$ should be equal to the variety obtained by Vasiu's 'shifting process', loc. cit. 6.4.

In section 3 we prove some properties comparing the fields of definition of an abelian variety and a weak Mumford–Tate lift. Using the theory of absolute Hodge motives, the above statements concerning Galois representations are then derived from the corresponding statements about the Hodge structures associated to weak Mumford–Tate liftings. The arguments are based on those used in [Noo01] and [Pau04].

In the above statements, the abelian varieties with Mumford–Tate group of type $D_k^{\mathbf{H}}$ stand out as an exception to the general situation. These varieties deserve a separate treatment, and this is the subject of section 5. Let F be a number field as above, A an abelian variety over F and assume that, for the Mumford–Tate group G_A, the derived group $G^{\mathrm{der}}_{A/\mathbf{C}}$ is isomorphic to a product $\prod G_{A,i}$ of h-maximal groups of type $D_k^{\mathbf{H}}$. It can be shown that such abelian varieties do indeed exist. It turns out that the associated system $(\rho_{A,p})$ of p-adic Galois representations admits a system of non-trivial weak geometric liftings.

The corollary 5.9 states that there exists a group $\widetilde{G}$ such that $\widetilde{G}^{\mathrm{der}} \to G_A^{\mathrm{der}}$ is the universal cover and, as always after replacing F by a finite extension, a system $(\tilde{\rho}_p)$ of weak geometric liftings of the $\rho_{A,p}$. These representations are called the p-adic Galois representations of *lifted abelian $D_k^{\mathbf{H}}$-type*. The construction makes use, again, of a lifting $\tilde{h}\colon S \to \widetilde{G}_{/\mathbf{R}}$ of $h\colon S \to G_{A/\mathbf{R}}$. This time, the existence of the system $(\tilde{\rho}_p)$ follows from [Win95].

When considering weak geometric liftings of Galois representations associated to abelian varieties, these representations of lifted abelian $D_k^{\mathbf{H}}$-type are the only 'new' representations that one may encounter. To be precise, let A/F be an abelian variety with Mumford–Tate group G_A. For any group G' and isogeny $G'^{\mathrm{der}} \to G_A^{\mathrm{der}}$ and any weak geometric lift $\rho'_p\colon \mathscr{G}_F \to G'(\mathbf{Q}_p)$ of $\rho_{A,p}$, the representation ρ'_p belongs to the tannakian category generated by the p-adic Galois representations associated to abelian varieties, the representations of lifted abelian $D_k^{\mathbf{H}}$-type and those with finite image, see corollary 5.12.

In the final section 6 we study the representations $\tilde{\rho}_p$ of lifted abelian $D_k^{\mathbf{H}}$-type. It is proved in 6.2 that such a representation $\tilde{\rho}_p$ generally does not belong to the tannakian category generated by the Galois representations associated to abelian varieties. The question if $\tilde{\rho}_p$ belongs to the tannakian category generated by the Galois representations associated to algebraic varieties, as predicted by the Fontaine–Mazur conjecture, remains open.

Combined with 'standard' conjectures, the conjecture of Fontaine and Mazur implies that the $\tilde{\rho}_p$ have certain 'motivic' properties. For example, any Frobenius element in $\mathscr{G}_F$ should act semi-simply on any representation $\widetilde{V}_p$ of $\widetilde{G}_{/\mathbf{Q}_p}$ and the eigenvalues of the image should be Weil numbers, i. e. algebraic integers of which all complex absolute values coincide. Several of these motivic properties are proved in the final part of section 6.

In addition to their relevance to the Fontaine–Mazur conjecture, the construction of weak geometric liftings has applications to the study of Galois representations associated to abelian varieties. We have seen that for an abelian variety A/F, and for F large enough, the representations $\rho_{A,p}\colon \mathscr{G}_F \to G_A(\mathbf{Q}_p)$ have weak geometric liftings $\rho_{B,p}\colon \mathscr{G}_F \to G_B(\mathbf{Q}_p)$, where B/F is an abelian variety and $G_B^{\mathrm{der}} \to G_A^{\mathrm{der}}$ is not far from the universal cover. The Galois representation on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ belongs to the tannakian category generated by $\mathrm{H}^1_{\text{ét}}(B_{\bar{F}}, \mathbf{Q}_p)$ and the representations associated to abelian varieties of CM-type. This can be useful because the representation $\mathrm{H}^1_{\text{ét}}(B_{\bar{F}}, \mathbf{Q}_p)$ is in general easier to study than

$\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$. A first example of such an application can be found in the paper [Pau04] of F. Paugam. In [Vas03], A. Vasiu uses his technique of adjoint isogenous abelian varieties to study new instances of the Mumford–Tate conjecture. Other applications are to follow in forthcoming publications.

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1 Preliminaries

1.1 Basic notations. For any field F, we denote by $\bar{F}$ an algebraic closure of F. The absolute Galois group of F is the group $\mathscr{G}_F = \mathrm{Aut}_F(\bar{F})$.

For any prime number p, the field $\mathbf{Q}_p$ is the p-adic completion of $\mathbf{Q}$ and $\mathbf{C}_p$ is the completion of an algebraic closure $\overline{\mathbf{Q}}_p$.

If G is a group and K a field, then $\mathbf{Rep}_K(G)$ is the category of finite dimensional K-linear representations of G. If G is a topological group (a Galois group for example) and K a topological field, then the representations in $\mathbf{Rep}_K(G)$ are assumed to be continuous. If G is a linear algebraic group, then $\mathbf{Rep}_K(G)$ is the category of algebraic representations of G.

If G is an algebraic group, a *quasi-cocharacter* of G is an element of the direct limit

$$\varinjlim_{k\in\mathbf{N}} \mathrm{Hom}(\mathbf{G}_m^{(k)}, G),$$

where the transition map $\mathbf{G}_m^{(k\ell)} \to \mathbf{G}_m^{(k)}$ is the morphism $z \mapsto z^\ell$. Giving a quasi-cocharacter of G is equivalent to giving an integer k and a cocharacter $\mu\colon \mathbf{G}_m \to G$. Intuitively the quasi-cocharacter given by (k, μ) is the kth root of μ.

1.2 Absolute Hodge motives. We will freely use the language of tannakian categories. Everything we need here can be found in [DM82] which is to be considered the authoritative reference for all notions used but not explained in this paper. In particular, if $\mathscr{C}$ is a tannakian category, then the subcategory

$\otimes$-generated by a collection X of objects of $\mathscr{C}$ is the smallest subcategory of $\mathscr{C}$ containing all objects which are isomorphic to a subquotient of a polynomial expression with coefficients in $\mathbf{N}$ in the objects contained in X. In such a polynomial expression, $+$ and $\cdot$ are to be interpreted as $\oplus$ and $\otimes$ respectively.

For any field F of characteristic 0, let $\mathbf{Mot}_{\mathrm{AH}}(F)$ be the category of motives for absolute Hodge cycles as described in [DM82], especially section 6 of that paper. It is constructed as Grothendieck's category of Chow motives except where it concerns the morphisms which are given by absolute Hodge classes, not by cycle classes as for Grothendieck motives. It must be pointed out that by definition, the morphisms between two motives M_1 and M_2 defined over F are the appropriate absolute Hodge classes on the product which are defined over F, i. e. Hodge classes on $(M_1 \times M_2)_{\bar{F}}$ invariant under the action of $\mathscr{G}_F$.

Motives are graded objects, each motive M is a finite direct sum $\bigoplus M^i$, where each M^i is a pure motive of weight i. The *Tate motive* $\mathbf{Q}(1)$ is the dual of $h^2(\mathbf{P}^1_F)$ in the category $\mathbf{Mot}_{\mathrm{AH}}(F)$.

1.3 Mumford–Tate groups. In what follows, we will assume that F is contained in $\mathbf{C}$ and we write $\bar{F}$ for the algebraic closure of F in $\mathbf{C}$. Assume that $M_1, \ldots, M_r$ are absolute Hodge motives over F, and let $\mathscr{C} = \langle M_1, \ldots, M_r, \mathbf{Q}(1) \rangle$ be the tannakian subcategory of $\mathbf{Mot}_{\mathrm{AH}}(F)$ which is $\otimes$-generated by $M_1, \ldots, M_r$ and the Tate motive $\mathbf{Q}(1)$. Let $\mathrm{H_B}(\cdot, \mathbf{Q})$ be the fibre functor of $\mathscr{C}$ over $\mathbf{Q}$ defined by the Betti cohomology $\mathrm{H}^*_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$ of complex algebraic varieties. Then the *Mumford–Tate group G of $\mathscr{C}$* is by definition the automorphism group of $\mathrm{H_B}(\cdot, \mathbf{Q})$. The connected component of G is reductive and G acts on $\mathrm{H_B}(M, \mathbf{Q})$ for every object M of $\mathscr{C}$. The fibre functor $\mathrm{H_B}(\cdot, \mathbf{Q})$ induces an equivalence between $\mathscr{C}$ and the category $\mathbf{Rep}_{\mathbf{Q}}(G)$. In particular, for $M, M' \in \mathscr{C}$, one has

$$\mathrm{Hom}_{\mathscr{C}}(M, M') = \mathrm{Hom}_G(\mathrm{H_B}(M, \mathbf{Q}), \mathrm{H_B}(M', \mathbf{Q})),$$

i. e. the action of G fixes all absolute Hodge classes defined over F on all objects of $\mathscr{C}$. Moreover, G is the smallest $\mathbf{Q}$-algebraic group with this property. Note that the connected component of G is the Mumford–Tate group of the subcategory of $\mathbf{Mot}_{\mathrm{AH}}(\bar{F})$ which is $\otimes$-generated by $M_{1/\bar{F}}, \ldots, M_{r/\bar{F}}$ and $\mathbf{Q}(1)$. When considering only one motive M, the Mumford–Tate group G_M of M is the Mumford–Tate group of the subcategory $\langle M, \mathbf{Q}(1) \rangle$ of $\mathbf{Mot}_{\mathrm{AH}}(F)$.

For a general subcategory $\mathscr{C}$ of $\mathbf{Mot}_{\mathrm{AH}}(F)$, we define the Mumford–Tate group in a similar way, obtaining a pro-algebraic group.

1.4 Subcategories of $\mathbf{Mot}_{\mathrm{AH}}$. An *Artin motive* is an object of $\mathbf{Mot}_{\mathrm{AH}}(F)$ with finite Mumford–Tate group. The category of Artin motives is the tannakian subcategory of $\mathbf{Mot}_{\mathrm{AH}}(F)$ generated by the finite F-schemes. An *abelian motive* over F is an AH-motive which belongs to the tannakian subcategory of $\mathbf{Mot}_{\mathrm{AH}}(F)$ generated by the motives of abelian varieties, the Tate motive and the Artin motives. We will write $(\mathbf{Artin})_F$ for the category of Artin motives and $(\mathbf{AV})_F$ for the category of abelian motives over F. An abelian variety is *potentially of CM-type* if the connected component of its Mumford–Tate group is commutative and $(\mathbf{CM})_F$ is the tannakian subcategory of $\mathbf{Mot}_{\mathrm{AH}}(F)$ generated by the Artin motives, $\mathbf{Q}(1)$ and the motives of the abelian varieties which are potentially of CM-type.

If A is an abelian variety over F, then $h(A) = \bigoplus_i \bigwedge^i h^1(A)$, so in most questions concerning abelian motives, it suffices to consider the motives $h^1(A)$ instead of the $h(A)$.

1.5 Betti realization. For any absolute Hodge motive M, the Betti realization $\mathrm{H}_{\mathrm{B}}(M, \mathbf{Q})$ carries a Hodge structure. Giving this Hodge structure is equivalent to giving an action of the group $S = \mathbf{C}^\times$, viewed as an algebraic group over $\mathbf{R}$, on the $\mathbf{R}$-vector space $\mathrm{H}_{\mathrm{B}}(M, \mathbf{Q}) \otimes \mathbf{R}$. As the the Mumford–Tate group G_M of M fixes all (absolute) Hodge classes, this action is given by a morphism of algebraic groups $h\colon S \to G_{M/\mathbf{R}}$. The couple (G_M, h) is the *Mumford–Tate datum* associated to M. More generally, if $\mathscr{C}$ is a $\otimes$-subcategory of $\mathbf{Mot}_{\mathrm{AH}}(F)$, with Mumford–Tate group G, then there is a morphism $h\colon S \to G_{/\mathbf{R}}$ which functorially defines the Hodge structures on $\mathrm{H}_{\mathrm{B}}(M, \mathbf{Q})$ for all objects M of $\mathscr{C}$.

We restrict our attention to the category of abelian motives $(\mathbf{AV})_F$. As explained in [DM82, 6.25], it follows from [Del82a, Theorem 2.11] that the Betti realization induces an equivalence of $(\mathbf{AV})_F$ with its essential image in the category **Hodge** of Hodge structures.

Let $\mathscr{C}$ be a $\otimes$-subcategory of $\mathbf{Mot}_{\mathrm{AH}}(F)$ and let (G, h) be the associated Mumford–Tate datum. For any object M of $\mathscr{C}$, the Betti realization $\mathrm{H}_{\mathrm{B}}(M, \mathbf{Q})$ is a representation of G and the Hodge structure on $\mathrm{H}_{\mathrm{B}}(M, \mathbf{Q})$ is given by the action of S on $\mathrm{H}_{\mathrm{B}}(M, \mathbf{Q}) \otimes \mathbf{R}$ induced by $h\colon S \to G_{/\mathbf{R}}$. Since every $\mathbf{Q}$-linear

representation of G is the Betti realization of an object of $\mathscr{C}$, this construction defines a $\otimes$-equivalence of $\mathbf{Rep}_{\mathbf{Q}}(G)$ with the essential image of $\mathscr{C}$ in **Hodge**.

If A is an abelian variety, its Mumford–Tate group G_A is equal to the Mumford–Tate group of $h^1(A)$. The connected component of G_A coincides in turn with the Mumford–Tate group of the Hodge structure $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$.

Let (G, h) be a Mumford–Tate datum. Composition of h with the cocharacter $\mathbf{G}_{m/\mathbf{C}} \to S_{/\mathbf{C}}$ dual to $z\colon S_{/\mathbf{C}} \to \mathbf{G}_{m/\mathbf{C}}$ gives rise to *Hodge cocharacter* $\mu\colon \mathbf{G}_{m/\mathbf{C}} \to G_{M/\mathbf{C}}$. Alternatively, μ is determined by the condition that $\mathbf{G}_{m/\mathbf{C}}$ acts on the factor $V^{p,q}$ of the Hodge decomposition as multiplication by z^p. Conversely, a pure Hodge structure is determined by giving its weight and the Hodge cocharacter.

1.6 ***p*-adic Galois representations.** Let p be a prime number. The étale cohomology with coefficients in $\mathbf{Q}_p$ of algebraic varieties over $\bar{F}$ defines the p-adic étale realization on the category of absolute Hodge motives. The p-adic étale realization of a motive M over F is a $\mathbf{Q}_p$-vector space $\mathrm{H}_{\text{ét}}(M_{\bar{F}}, \mathbf{Q}_p)$ endowed with a continuous action of the group $\mathscr{G}_F$. It follows from standard tannakian theory that the image of $\mathscr{G}_F$ in $\mathrm{GL}(\mathrm{H}_{\text{ét}}(M_{\bar{F}}, \mathbf{Q}_p))$ lies in $G_M(\mathbf{Q}_p)$, where G_M is the Mumford–Tate group of M. The representation thus gives rise to a morphism $\rho_{M,p}\colon \mathscr{G}_F \to G_M(\mathbf{Q}_p)$.

For any prime number p, let $(\mathbf{Artin})$-$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ be the tannakian subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ generated by the p-adic étale realizations of the objects of $(\mathbf{Artin})_F$. This category coincides with the category of $\mathbf{Q}_p$-linear representations of $\mathscr{G}_F$ with finite image. Similarly, we let let $(\mathbf{AV})$-$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ be the tannakian subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ generated by the p-adic étale realizations of the objects of $(\mathbf{AV})_F$. As before, in the case of an abelian variety A, it is usually sufficient to consider just the representation on the first étale cohomology group $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$.

Restricting to an inertia group at a p-adic place of F, these representations give rise to representations of *Hodge–Tate type*, cf. [Fon94, §3] or [Ser78]. If $\mathscr{I} \subset \mathscr{G}_F$ is such an inertia subgroup and V_p is a $\mathbf{Q}_p$-linear representation of $\mathscr{I}$ of Hodge–Tate type, there is a canonical decomposition

$$V_p \otimes_{\mathbf{Q}_p} \mathbf{C}_p = \bigoplus_{p,q} V^{p,q},$$

the *Hodge–Tate decomposition* of $V_p \otimes \mathbf{C}_p$. The *Hodge–Tate cocharacter* is the cocharacter $\mu\colon \mathbf{G}_{m/\mathbf{C}_p} \to \mathrm{GL}(V_p \otimes \mathbf{C}_p)$ such that action of $\mathbf{G}_{m/\mathbf{C}_p}$ on $V^{p,q}$ is the multiplication by x^p. It can be shown (see [Ser78, §1]) that the connected component H of the Zariski closure of the image of $\mathscr{I}$ in $\mathrm{GL}(V_p)$ coincides with the smallest subgroup $H \subset \mathrm{GL}(V_p)$ such that μ factors through $H_{/\mathbf{C}_p}$.

2 Mumford–Tate liftings of abelian varieties over C

2.1 Mumford–Tate liftings. Let $M/\mathbf{C}$ be an abelian motive, G_M its Mumford–Tate group and (G_M, h_M) the associated Mumford–Tate datum. This means that the Hodge structure on $V_M = \mathrm{H}_\mathrm{B}(M, \mathbf{Q})$ is defined by $h_M\colon S \to G_{M/\mathbf{R}}$.

We say that an abelian motive N with Mumford–Tate datum (G_N, h_N) provides a *Mumford–Tate lift* of M if there exists a central isogeny $\pi\colon G_N \to G_M$ such that $\pi_\mathbf{R} \circ h_N = h_M$. We say that M is *Mumford–Tate liftable* if there exists an abelian motive $N/\mathbf{C}$ giving a Mumford–Tate lift of M and such the morphism $\pi\colon G_N \to G_M$ is not an isomorphism. We say that M is *Mumford–Tate unliftable* if it is not Mumford–Tate liftable.

If there exists a central isogeny $\pi^\mathrm{der}\colon G_N^\mathrm{der} \to G_M^\mathrm{der}$ such that

$$\pi_N^\mathrm{ad} \circ h_N = \pi_M^\mathrm{ad} \circ h_M$$

then N provides a *weak Mumford–Tate lift* of M. Here the maps π_M^ad and π_N^ad are the projections $G_M \to G_M^\mathrm{ad}$ and $G_N \to G_N^\mathrm{ad}$. As it is a central isogeny, it follows that π^der induces an isomorphism $G_M^\mathrm{ad} \cong G_N^\mathrm{ad}$, giving a sense to the equality $\pi_N^\mathrm{ad} \circ h_N = \pi_M^\mathrm{ad} \circ h_M$. Finally, M is *essentially Mumford–Tate unliftable* if there does not exists any abelian motive $N/\mathbf{C}$ giving a weak Mumford–Tate lift of M for which π^der is not an isomorphism.

We will often write M-T (un)liftable instead of Mumford–Tate (un)liftable.

2.2 Remark. If N is a weak Mumford–Tate lift of M, then M and N are adjoint-isogenous in the sense of [Vas03, 6.1] and conversely, if M and N are adjoint-isogenous and if there is an isogeny $G_N^\mathrm{der} \to G_M^\mathrm{der}$, then N is a weak Mumford–Tate lift of M.

2.3 The Mumford–Tate datum of an abelian variety. Let $A/\mathbf{C}$ be an abelian variety, (G_A, h) the associated Mumford–Tate datum and $\mu\colon \mathbf{G}_{m/\mathbf{C}} \to G_{A/\mathbf{C}}$ be

the Hodge cocharacter. It follows from [Del79], in particular from 1.3 and 2.3, that the simple factors of $G^{\mathrm{ad}}_{A/\mathbf{R}}$ are absolutely simple and of classical type (A, B, C or D). The group $\mathscr{G}_{\mathbf{Q}}$ acts on the set of these simple factors and each factor is conjugate to a non-compact one. The compact factors are exactly the factors to which h projects trivially.

Let H be a $\mathbf{Q}$-simple factor G^{ad}_{A} and decompose $H_{/\mathbf{C}} = \prod_{\iota \in I_H} H_{/\mathbf{C},i}$ for some finite set I_H with $\mathscr{G}_{\mathbf{Q}}$-action. Here we have identified $\overline{\mathbf{Q}}$ with the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. Note that the complex conjugation in $\mathscr{G}_{\mathbf{Q}}$ acts trivially on I because the simple factors of $G^{\mathrm{ad}}_{A/\mathbf{R}}$ are absolutely simple. Let $I_{H,c}$ be the set of indices such that the Hodge cocharacter μ projects trivially to $H_{/\mathbf{C},\iota}$ and let $I_{H,nc}$ be the complement of $I_{H,c}$ in I_H. The $\iota \in I_{H,c}$ are exactly the indices for which the corresponding real factor of $H_{/\mathbf{R}}$ is compact. For each $\iota \in I_{H,nc}$, the Hodge cocharacter lifts to a quasi-cocharacter $\tilde{\mu}_\iota$ of the universal cover $\widetilde{H}_{/\mathbf{C},\iota}$ of $H_{/\mathbf{C},\iota}$. If the simple factors of $H_{/\mathbf{C}}$ are of type A, B or C then it follows from [Del79, 1.3] that $\widetilde{H}_{/\mathbf{C},\iota}$ admits a faithful representation $W_{/\mathbf{C},\iota}$ such that $\tilde{\mu}_\iota$ acts on $W_{/\mathbf{C},\iota}$ with exactly two weights. Contemplating the tables [Del79, 1.3.9] or [Pin98, Table 4.2] one sees that the highest weight of $W_{/\mathbf{C},\iota}$ is

- either ϖ_1 or ϖ_k if the simple factors of $H_{/\mathbf{C}}$ are of type A_k,
- ϖ_k if these simple factors are of type B_k and
- ϖ_1 if the factors are of type C_k.

For each $\iota \in I_H$, we define a representation $V_{/\mathbf{C},\iota}$ of $\widetilde{H}_{/\mathbf{C},\iota}$ as follows. Let $V_{/\mathbf{C},\iota}$ be the direct sum of the representations with highest weights ϖ_1 and ϖ_k if H is of type A_k with $k \geq 2$ and define $V_{/\mathbf{C},\iota}$ to be the representation with highest weight ϖ_1 (resp. ϖ_k) if H is of type A_1 or C_k (resp. B_k). For $\iota \in I_{H,nc}$, the cocharacter $\tilde{\mu}_\iota$ still acts on each irreducible factor of $V_{/\mathbf{C},\iota}$ with exactly two weights. The product over $\iota \in I_H$ of the $\widetilde{H}_{/\mathbf{C},\iota}$ descends to an algebraic group $\widetilde{H}$ over $\mathbf{Q}$ and a multiple of the direct sum of the $V_{/\mathbf{C},\iota}$ descends to a faithful $\mathbf{Q}$-linear representation V of $\widetilde{H}$.

2.4 Remarks. If the simple factors of $H_{/\mathbf{C}}$ are of type B_k or C_k then the condition that $\tilde{\mu}_\iota$ acts on $W_{/\mathbf{C},\iota}$ with exactly two weights uniquely determines the highest weight. Similarly, if the simple factors of $H_{/\mathbf{C}}$ are of type A_k and if $\tilde{\mu}_\iota$ is not

dual to α_1, only the representations with highest weight ϖ_1 or ϖ_k fulfill this condition. Obviously, all these representations are faithful.

In the case where the simple factors of $H_{/\mathbf{C}}$ are of type A_k and where $\tilde{\mu}_\iota$ is dual to α_1, the cocharacter $\tilde{\mu}_\iota$ acts with exactly two weights on the representation with highest weight ϖ_s for any $1 \leq s \leq k$, see [Pin98, Table 4.2]. Only for $s = 1$ and for $s = k$ does one obtain a faithful representation.

Also note that $V_{/\mathbf{C},\iota}$ is self dual in each case, see the aforementioned table in Pink's paper.

2.5 Factors of type D_k. In what follows, we will focus on the factors of G_A^{ad} of type D_k, first considering the case $k \geq 5$. Fix a $\mathbf{Q}$-simple factor H of G_A^{ad} such that the simple factors of $H_{/\mathbf{C}}$ are of type D_k with $k \geq 5$. Let $H_{/\mathbf{C}} = \prod_{\iota \in I_H} H_{/\mathbf{C},\iota}$ and let $I_{H,c}$ and $I_{H,nc}$ be as before. The Dynkin diagram of $H_{/\mathbf{C}}$ is the disjoint union, indexed by I_H, of diagrams of type D_k, and it follows from [Del79, 1.3] that, for each $\iota \in I_{H,nc}$, the conjugacy class of the cocharacter μ_ι is dual to an endpoint $\alpha^{(\iota)}$ of the ι-component of the Dynkin diagram.

2.6 Lemma. *Assume that $k \geq 5$ and that $\alpha^{(\iota)} = \alpha_1$ for some $\iota \in I_{H,nc}$. Then $\alpha^{(\kappa)} = \alpha_1$ for all $\kappa \in I_{H,nc}$.*

Proof. Assume that $\alpha^{(\iota)} = \alpha_1$ for $\iota \in I_{H,nc}$ and let $G^{\mathrm{der}}_{A/\mathbf{C},\iota}$ be the simple factor of $G^{\mathrm{der}}_{A/\mathbf{C}}$ projecting onto $H_{/\mathbf{C},\iota}$. Via the representation of G_A on $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$, the Hodge cocharacter μ acts on $V_{/\mathbf{C}}$ with two weights. By the table in [Del79, 1.3], this implies that every non-trivial irreducible direct factor of the representation of $G^{\mathrm{der}}_{A/\mathbf{C},\iota}$ on $V_{/\mathbf{C}}$ has highest weight ϖ_{k-1} or ϖ_k. As H is $\mathbf{Q}$-simple, the same thing is true for the representation on $V_{/\mathbf{C}}$ of the simple factors of $G^{\mathrm{der}}_{A/\mathbf{C}}$ mapping to the factors $H_{/\mathbf{C},\kappa}$ for the other $\kappa \in I_H$. Using the tables [Del79, 1.3.9], the fact that μ acts with two weights implies that all non-trivial μ_κ are dual to α_1. ∎

If any factor of $H_{/\mathbf{C}}$ satisfies the conditions of the lemma, then we say that H and the factors of $H_{/\mathbf{R}}$ and $H_{/\mathbf{C}}$ are *of type* $D_k^{\mathbf{R}}$. As H was assumed to be $\mathbf{Q}$-simple, the vertices $\alpha_1^{(\iota)}$ form a single orbit for the action of $\mathscr{G}_{\overline{\mathbf{Q}}}$ in this case. In the opposite case we say that they are *of type* $D_k^{\mathbf{H}}$. In the latter case, the conjugacy class of the projection of μ to any factor of $H_{/\mathbf{C}}$ is either trivial or dual to α_{k-1} or α_k.

Let $H_{/\mathbf{C},\iota}$ be a factor of $G^{\mathrm{ad}}_{A/\mathbf{C}}$ of type $D^{\mathbf{R}}_k$ and let $\widetilde{H}_{/\mathbf{C},\iota}$ be its universal cover. For each $\iota \in I_H$, let $V_{/\mathbf{C},\iota}$ be the direct sum of the representations of $H_{/\mathbf{C},\iota}$ with highest weights ϖ_{k-1} and ϖ_k. As above, for $\iota \in I_{H,nc}$, the projection of the Hodge cocharacter μ to $H_{/\mathbf{C},\iota}$ lifts to a quasi-cocharacter $\tilde{\mu}_\iota$ of $\widetilde{H}_{/\mathbf{C},\iota}$ and $\tilde{\mu}_\iota$ acts on $V_{/\mathbf{C},\iota}$ with two weights $\pm 1/2$. The product over $\iota \in I$ of the $\widetilde{H}_{/\mathbf{C},\iota}$ and a multiple of the direct sum of the $V_{/\mathbf{C},\iota}$ descend to an algebraic group $\widetilde{H}$ over $\mathbf{Q}$ and a faithful $\mathbf{Q}$-linear representation V of $\widetilde{H}$.

Next assume that $H_{/\mathbf{C},\iota}$ is a factor of $G^{\mathrm{ad}}_{A/\mathbf{C}}$ of type $D^{\mathbf{H}}_k$. As before, for $\iota \in I_{H,nc}$, the projection of the Hodge cocharacter lifts to a quasi-cocharacter $\tilde{\mu}_\iota$ of the universal cover $\widetilde{H}_{/\mathbf{C},\iota}$, but this time the group $\widetilde{H}_{/\mathbf{C},\iota}$ does not have a faithful representation on which $\tilde{\mu}_\iota$ acts with two weights. This property can only be achieved for a quotient of $\widetilde{H}_{/\mathbf{C},\iota}$ which can be constructed as follows. For each $\iota \in I_H$, let $V_{/\mathbf{C},\iota}$ be the representation of $\widetilde{H}_{/\mathbf{C},\iota}$ with highest weight ϖ_1. We will refer to the quotient of $\widetilde{H}_{/\mathbf{C},\iota}$ which acts faithfully on $V_{/\mathbf{C},\iota}$ as the *h-maximal cover* of $H_{/\mathbf{C},\iota}$. This h-maximal cover descends over $\mathbf{R}$ to an inner form of the orthogonal group $\mathrm{SO}(2k)$. For each $\iota \in I_{H,nc}$, the quasi-cocharacter $\tilde{\mu}_\iota$ acts on $V_{/\mathbf{C},\iota}$ with weights $\pm 1/2$. The product over $\iota \in I$ of these h-maximal covers descends to an algebraic group over $\mathbf{Q}$ and a multiple of the direct sum of the $V_{/\mathbf{C},\iota}$ descends to a faithful $\mathbf{Q}$-linear representation of this group.

2.7 Factors of type D_4. We finally consider the case where $k = 4$, so let H be a $\mathbf{Q}$-simple factor of G^{ad}_A such that the simple factors of $H_{/\mathbf{C}}$ are of type D_4. In this case, the automorphism group of the Dynkin diagram permutes the set of endpoints, so the types $D^{\mathbf{R}}_4$ and $D^{\mathbf{H}}_4$ can not be distinguished in the same manner as before. The fundamental difference between the two types is the existence of a $\mathbf{Q}$-linear representation V as above. We formalize this as follows.

Let $H_{/\mathbf{C}} = \prod_{\iota \in I_H} H_{/\mathbf{C},\iota}$ and $I_{H,nc}$ and $I_{H,c}$ as in the general case. For each $\iota \in I_{H,nc}$ let $\mu_\iota \colon \mathbf{G}_{m/\mathbf{C}} \to H_{/\mathbf{C},\iota}$ be the projection of the Hodge cocharacter. The conjugacy class of each μ_ι is dual to an endpoint of the corresponding component of the Dynkin diagram. Let Δ be the set of these vertices, it contains exactly one endpoint of the ι-component of the Dynkin diagram if $\iota \in I_{H,nc}$ and no vertices in the other components. Choose $\iota \in I_{H,nc}$, let $G^{\mathrm{der}}_{A/\mathbf{C},\iota}$ be the almost simple factor of $G^{\mathrm{der}}_{A/\mathbf{C}}$ projecting onto $H_{/\mathbf{C},\iota}$ and let ϖ be the highest weight of some non trivial factor of the representation of $G^{\mathrm{der}}_{A/\mathbf{C},\iota}$ on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$. As the Hodge cocharacter acts on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{C})$ with two weights, the arguments of [Del79,

1.3, 2.3] imply that ϖ corresponds to an endpoint α of the Dynkin diagram of H and that the $\mathscr{G}_{\mathbf{Q}}$-orbit Δ' of α does not meet Δ. In particular, there exists a $\mathscr{G}_{\mathbf{Q}}$-stable set Δ' of endpoints such that $\Delta \cap \Delta' = \emptyset$. The set Δ' contains at least one endpoint in each connected component of the Dynkin diagram.

If there exists a $\mathscr{G}_{\mathbf{Q}}$-stable set $\Delta'_{\max}$ of endpoints of the Dynkin diagram with $\Delta \cap \Delta'_{\max} = \emptyset$ containing two endpoints in each connected component, then we say that H and the factors of $H_{/\mathbf{R}}$ and $H_{/\mathbf{C}}$ are *of type* $D_4^{\mathbf{R}}$. If there does not exist such a $\Delta'_{\max}$, they are *of type* $D_4^{\mathbf{H}}$.

Assume that H is a factor of $G^{\mathrm{ad}}_{A/\mathbf{C}}$ of type $D_4^{\mathbf{R}}$, and let $\iota \in I_H$. Then the ι-component of the Dynkin diagram contains two endpoints α_ι and $\beta_\iota \in \Delta'_{\max}$. Let $V_{/\mathbf{C},\iota}$ be the direct sum of the representations of the universal cover $\widetilde{H}_{/\mathbf{C},\iota}$ with highest weights corresponding to α_ι and β_ι respectively. It is a faithful representation of $\widetilde{H}_{/\mathbf{C},\iota}$ and as $\alpha_\iota, \beta_\iota \notin \Delta$, it follows from [Del79, 1.3.9] that, for $\iota \in I_{H,nc}$, the lifting $\tilde{\mu}_\iota$ of μ_ι to $\widetilde{H}_{/\mathbf{C},\iota}$ acts on $V_{/\mathbf{C},\iota}$ with weights $\pm 1/2$. The product of the $\widetilde{H}_{/\mathbf{C},\iota}$ and a multiple of the direct sum of the $V_{/\mathbf{C},\iota}$ descend to a $\mathbf{Q}$-algebraic group $\widetilde{H}$ with a faithful $\mathbf{Q}$-linear representation V.

If $H_{/\mathbf{C},\iota}$ is a factor of $G^{\mathrm{ad}}_{A/\mathbf{C}}$ of type $D_4^{\mathbf{H}}$, then the above set Δ' meets the Dynkin diagram of $H_{/\mathbf{C},\iota}$ in one endpoint, corresponding to a fundamental weight ϖ. The *h-maximal cover* of $H_{/\mathbf{C},\iota}$ is the quotient of its universal cover $\widetilde{H}_{/\mathbf{C},\iota}$ acting faithfully on the representation of $\widetilde{H}_{/\mathbf{C},\iota}$ with highest weight ϖ. As in the case $k \geq 5$, the product of these data over all $\iota \in I_H$ descends to an algebraic group over $\mathbf{Q}$ together with a faithful $\mathbf{Q}$-linear representation. This cover of a $\mathbf{Q}$-simple factor of G^{ad}_A of type $D_4^{\mathbf{H}}$ is the h-maximal cover of G^{ad}_A.

2.8 Remark. For all factors $H_{/\mathbf{C},\iota}$ of type D, the representation $V_{/\mathbf{C},\iota}$ is self dual. As in the case of the factors of types A, B and C this can be read off from [Pin98, Table 4.2].

2.9 Theorem. *Let $A/\mathbf{C}$ be an abelian variety and let (G_A, h) be the associated Mumford–Tate datum. Then the following conditions are equivalent.*

1. *A is essentially Mumford–Tate unliftable.*
2. *The group $G^{\mathrm{der}}_{A/\mathbf{C}}$ is the product of its simple factors. The simple factors of types A_k, B_k, C_k and $D_k^{\mathbf{R}}$ are simply connected and the factors of type $D_k^{\mathbf{H}}$ are h-maximal in the sense defined above.*

2.10 Definition. Let $A/\mathbf{C}$ be an abelian variety, G_A its Mumford–Tate group and $V_A = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$. If G_A^{ad} is $\mathbf{Q}$-simple, then we say that A is *Mumford–Tate decomposed* if the following conditions hold.

- There are a totally real field K_0 and an absolutely simple algebraic group G^s over K_0 such that $G_A^{\mathrm{der}} \cong \mathrm{Res}_{K_0/\mathbf{Q}} G^s$.
- There is a faithful representation V^s of G^s such that the representation of G_A^{der} on V is isomorphic to $\mathrm{Res}_{K_0/\mathbf{Q}} V^s$.
- There is no proper non-trivial abelian subvariety B of A with Mumford–Tate group G_B verifying $G_B^{\mathrm{der}} = G_A^{\mathrm{der}}$.

We say that an abelian variety $A/\mathbf{C}$ is *Mumford–Tate decomposed* if it is isogenous to a product $\prod A_i$ such that each A_i is Mumford–Tate decomposed with $G_{A_i}^{\mathrm{ad}}$ simple and if G_A^{ad} is the product of the $G_{A_i}^{\mathrm{ad}}$. An abelian variety over a number field $F \subset \mathbf{C}$ is *Mumford–Tate decomposed* if $A_{/\mathbf{C}}$ is Mumford–Tate decomposed.

2.11 Remark. The upshot of this definition is that the abelian varieties arising from the construction of [Del79, 2.3] are Mumford–Tate decomposed. The representation of G_A^{der} on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ is a direct sum of the representations of the $G_{A_i}^{\mathrm{der}}$ constructed in 2.3 and 2.5. See also the proofs given below.

The notion of being (essentially) Mumford–Tate unliftable is a condition on the Mumford–Tate datum of an abelian variety whereas the notion of being Mumford–Tate decomposed pertains to the action of the Mumford–Tate group on the first Betti cohomology group.

2.12 Theorem. *For every abelian variety $A/\mathbf{C}$ there exists a weak Mumford–Tate lift $B/\mathbf{C}$ of A such that B is essentially Mumford–Tate unliftable and Mumford–Tate decomposed.*

Proofs of 2.9 and 2.12. These results can be derived from the work of Satake, see for example [Del79]. The same argument can be found in [Vas03], see §4 and paragraphs 6.3 and 6.4 in particular. The strategy of the proof is as follows. It is first shown that the condition 2.9.2 implies the condition 2.9.1. We then prove that any abelian variety admits a weak Mumford–Tate lift satisfying 2.9.2 and which is Mumford–Tate decomposed. Thanks to the fact that 2.9.2 implies 2.9.1,

this M-T lift is also M-T unliftable. Finally, the proof that the condition 2.9.1 implies 2.9.2 is a formality.

First assume that A verifies the conditions of 2.9.2. We will show that it is essentially M-T unliftable, so let B be a weak Mumford–Tate lift of A and let G_B be its Mumford–Tate group. It has to be proved that the map $G_B^{\mathrm{der}} \to G_A^{\mathrm{der}}$ is an isomorphism. It suffices to prove this after extension of scalars to $\mathbf{C}$ and as the only non-simply connected factors of $G_{A/\mathbf{C}}^{\mathrm{der}}$ are the factors of type $D_k^{\mathbf{H}}$, we only need to consider these factors.

Consider a factor $H_{/\mathbf{C}}$ of $G_{A/\mathbf{C}}^{\mathrm{der}}$ of type $D_k^{\mathbf{H}}$ to which the Hodge cocharacter projects non-trivially, assuming at first that $k \geq 5$. As we saw, the conjugacy class of the projection of the Hodge cocharacter to $H_{/\mathbf{C}}$ is dual to one of the vertices α_{k-1} or α_k of the Dynkin diagram. Let $\widetilde{H}_{/\mathbf{C}}$ be the factor of $G_{B/\mathbf{C}}^{\mathrm{der}}$ mapping onto $H_{/\mathbf{C}}$. An appropriate direct factor W of the representation of $\widetilde{H}_{/\mathbf{C}}$ on $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q}) \otimes \mathbf{C}$ provides a faithful representation on which $\tilde{\mu}$ acts with two weights $\pm 1/2$. As explained in 2.5, it follows from the tables in [Del79, 1.3.9], that the highest weight of every irreducible direct factor of W is ϖ_1 and hence that $\widetilde{H}_{/\mathbf{C}}$ is isomorphic to $H_{/\mathbf{C}}$. This proves that 2.9.2 implies 2.9.1 if $k \geq 5$.

In the case where $k = 4$, first note that G_A^{der} is the product of its $\mathbf{Q}$-simple factors. Let H be a simple factor of type $D_4^{\mathbf{H}}$ and let $\widetilde{H}$ be the factor of G_B mapping onto H. Consider $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q})$ as a representation of $\widetilde{H}$ and let W be a direct factor which is a faithful representation of $\widetilde{H}$. Let $\widetilde{H}_{/\mathbf{C},\iota}$ be any factor of $\widetilde{H}_{/\mathbf{C}}$ and let $W_{/\mathbf{C},\iota}$ be any irreducible direct factor of the restriction of $W \otimes \mathbf{C}$ to $\widetilde{H}_{/\mathbf{C},\iota}$. Then the lifting to $\widetilde{H}_{/\mathbf{C},\iota}$ of the Hodge cocharacter either acts trivially on $W_{/\mathbf{C},\iota}$ or with exactly two weights. It follows from 2.7 that $\widetilde{H}$ is the h-maximal cover of H and as H was h-maximal by hypothesis it follows that $\widetilde{H} \cong H$. This proves that 2.9.2 implies 2.9.1 in case $k = 4$.

We next show that if A is any abelian variety, then there exists a weak Mumford–Tate lift B of A with Mumford–Tate group G_B satisfying the conditions of 2.9.2 and which is Mumford–Tate decomposed. This fact readily follows from [Del79, 2.3] and the discussions in 2.3, 2.5 and 2.7, we recall the argument.

Fix a $\mathbf{Q}$-simple factor H^{ad} of G_A^{ad}. It is of the form $\mathrm{Res}_{K_0/\mathbf{Q}} H^{s,\mathrm{ad}}$ for some totally real number field K_0 and some absolutely simple adjoint group $H^{s,\mathrm{ad}}$ over K_0. As usual, decompose $H^{\mathrm{ad}}_{/\mathbf{C}} = \prod_{\iota \in I_H} H_{/\mathbf{C},\iota}$, where I_H, $I_{H,nc}$ and $I_{H,c}$ are as before.

Unless $H^{\mathrm{ad}}_{/\mathbf{C},\iota}$ is of type $D_k^{\mathbf{H}}$, we let $\widetilde{H}^{s,\mathrm{der}}$ be the simply connected cover of

$H^{s,\mathrm{ad}}$. In the remaining case we let $\widetilde{H}^{s,\mathrm{der}}$ be the h-maximal cover of $H^{s,\mathrm{ad}}$, in the sense of 2.5 resp. 2.7. In all cases, put

$$\widetilde{H}^{\mathrm{der}} = \mathrm{Res}_{K_0/\mathbf{Q}} \widetilde{H}^{s,\mathrm{der}}.$$

For each $\iota \in I_{H,nc}$, the projection of the Hodge cocharacter to $H^{\mathrm{ad}}_{/\mathbf{C},\iota}$ lifts to a quasi-cocharacter $\tilde{\mu}_\iota$ of $\widetilde{H}^{\mathrm{der}}_{/\mathbf{C},\iota}$. The faithful representation V of $\widetilde{H}^{\mathrm{der}}$ constructed in 2.3 and 2.5 resp. 2.7 is of the form $V = \mathrm{Res}_{K_0/\mathbf{Q}} V^s$ for some faithful $\mathbf{Q}$-linear representation V^s of $\widetilde{H}^{s,\mathrm{der}}$. These data have the following properties.

- There is a map $\widetilde{H}^{\mathrm{der}} \to G_A^{\mathrm{der}}$ such that $\tilde{\mu} = \prod_{\iota \in I_{H,nc}} \tilde{\mu}_\iota$ lifts the Hodge cocharacter.
- If $\iota \in I_{H,nc}$, and if W is an irreducible factor of $V_{\mathbf{C},\iota}$, then the cocharacter $\tilde{\mu}$ acts either trivially on W or with two rational weights r and $r+1$.

There exist a torus T over $\mathbf{Q}$ acting $\widetilde{H}^{\mathrm{der}}$-linearly on V and a quasi-cocharacter μ_T of $T_{/\mathbf{C}}$ such that the product $\tilde{\mu}\mu_T$ acts trivially on the $V_{\mathbf{C},\iota}$ for $\iota \in I_{H,c}$ and with weights $\pm 1/2$ for $\iota \in I_{H,nc}$. In fact, T is characterized by the condition that $T_{/\mathbf{C}}$ is the group of the automorphisms of $V \otimes \mathbf{C}$ acting by scalar multiplication on each isobaric component of $V \otimes \mathbf{C}$ as $\widetilde{H}^{\mathrm{der}}_{/\mathbf{C}}$-representation. The existence of μ_T follows from the fact that if $\tilde{\mu}$ acts non-trivially on an isobaric component then it acts with two weights r and $r+1$. Let H' be the image of $\widetilde{H}^{\mathrm{der}} \times T$ in $\mathrm{GL}(V)$ and let $\mu' = \tilde{\mu}\mu_T$.

We choose a quadratic and totally imaginary extension L of K_0 and consider the $\mathbf{Q}$-algebraic group $L^\times$. The natural action of $L^\times$ on L gives rise to a $\mathbf{Q}$-linear representation W of $L^\times$. One has

$$\left(L^\times\right)_{/\mathbf{C}} \cong \bigoplus_{\iota \in I} \mathbf{G}^2_{m/\mathbf{C}}$$

and one defines a quasi-cocharacter ν of $\left(L^\times\right)_{/\mathbf{C}}$ by $\nu_\iota(z) = (z^{1/2}, z^{1/2})$ for $\iota \in I_{nc}$ and $\nu_\iota(z) = (z, 1)$ for $\iota \in I_c$. The action of $H' \times L^\times$ on $V \otimes_{K_0} W$ defines a faithful representation of a quotient $\widetilde{H}$ of $H' \times L^\times$ with derived group $\widetilde{H}^{\mathrm{der}}$ in which $(\tilde{\mu}, \nu)$ acts with weights 0 and 1. Let $\tilde{h} \colon S \to \widetilde{H}_{\mathbf{R}}$ be defined by

$$\tilde{h}(z, \bar{z}) = (\mu'\nu)(z)\overline{(\mu'\nu)(\bar{z})}.$$

Shrinking the centre of $\widetilde{H}$, we may assume that the image of $\tilde{h}$ is Zariski dense in $\widetilde{H}$. We claim that this defines the Shimura datum $(\widetilde{H}, \tilde{h})$ associated to an abelian

variety B_1. To see why this is the case, note that $\tilde{h}$ defines a $\mathbf{Q}$-Hodge structure of type $(1,0),(0,1)$ on $V \otimes_{K_0} W$. In order to establish that this Hodge structure comes from an abelian variety it is sufficient to show that it is polarizable, cf. [Del72, 2.3]. First use that, by loc. cit. 2.11, the element $\mathrm{ad}(h_A(i))$ defines a Cartan involution of G_A^{der} and hence that $\mathrm{ad}(\tilde{h}(i))$ is a Cartan involution of $\widetilde{H}^{\mathrm{der}}_{/\mathbf{R}}$. Next, one checks that the weight $w_h \colon \mathbf{G}_m \to \widetilde{H}$ is central, defined over $\mathbf{Q}$ and that the quotient of the center of $\widetilde{H}$ by $w_h(\mathbf{G}_m)$ is compact. The last statement is deduced from the fact that this center is contained in a product of CM tori. It now follows that $\mathrm{ad}(\tilde{h}(i))$ is a Cartan involution of $\widetilde{H}/w(\mathbf{G}_m)$ and by [Del79, 1.1.18(b)] this implies that the Hodge structure on $V \otimes_{K_0} W$ is polarizable.

The group $\widetilde{H}^{\mathrm{ad}}$ is $\mathbf{Q}$-simple, $\widetilde{H}^{\mathrm{der}}$ is its h-maximal cover and B_1 is Mumford–Tate decomposed by construction. Since G_A is the Mumford–Tate group of an abelian variety, G_A^{der} is a quotient of the h-maximal cover of G_A^{ad}, so there is an isogeny $\widetilde{H}^{\mathrm{der}} \to G_A^{\mathrm{der}}$ lifting $\widetilde{H}^{\mathrm{ad}} \to G_A^{\mathrm{ad}}$.

Applying this to all $\mathbf{Q}$-simple factors of G_A^{ad} we obtain Mumford–Tate decomposed abelian varieties B_i such that $B = \prod B_i$ is a weak M-T lift of A verifying the conditions of 2.9.2. By construction, if G_B (resp. G_{B_i}) denotes the Mumford–Tate group of B (resp. B_i), then $G_B^{\mathrm{der}} = \prod G_{B_i}^{\mathrm{der}}$. The first part of this proof implies that B is essentially M-T unliftable. This terminates the proof of theorem 2.12.

Finally, let A be essentially M-T unliftable, i. e. the condition 2.9.1 is satisfied. The theorem 2.12 implies that A has a weak M-T lift B with Mumford–Tate group G_B satisfying the condition of 2.9.2. As A is essentially M-T unliftable, we must have $G_A^{\mathrm{der}} \cong G_B^{\mathrm{der}}$, which implies that G_A also verifies 2.9.2. □

2.13 Remarks.

2.13.1 In the above proof, fix a $\mathbf{Q}$-simple factor H^{ad} of G_A^{ad}, let $\iota \in I_H$ and consider an irreducible factor W of the representation $V_{\mathbf{C},\iota}$ of $\widetilde{H}^{\mathrm{der}}_{/\mathbf{C},\iota}$. It follows from [Del79, 1.3] that the highest weight of W is a fundamental weight of $\widetilde{H}^{\mathrm{der}}_{/\mathbf{C},\iota}$. More precisely, according to the type of $\widetilde{H}^{\mathrm{der}}_{/\mathbf{C},\iota}$ and the quasi-cocharacter $\tilde{\mu}_\iota$, it is the weight given by the tables 1.3.9 of Deligne's paper or [Pin98, Table 4.2].

2.13.2 With the same notations, assume that H^{ad} is not of type A_k with $k \geq 2$. In the above construction of the essentially M-T unliftable variety B_i corresponding to this factor, we then have $r = 1/2$. This means that the construction of the

intermediate group H' can be shunted in this case. It follows that the Mumford–Tate group of B is contained in the image of $\widetilde{H}^{\mathrm{der}} \times L^\times$ in $\mathrm{GL}(V \otimes_{K_0} W)$.

This argument is also valid for factors of type A_k for which all the numbers r are equal to $1/2$.

2.13.3 Instead of using [Del72] to prove the fact $\tilde{h}$ defines a polarizable Hodge structure on $V \otimes_{K_0} W$, one may also explicitly construct a polarization. This is not difficult, using the autoduality of V as representation of $\widetilde{H}^{\mathrm{der}}$.

2.14 Examples.

2.14.1 Let $A/\mathbf{C}$ be an abelian variety arising from Mumford's construction, see [Mum69]. In this case one has $G^{\mathrm{der}}_{A/\overline{\mathbf{Q}}} \cong \mathrm{SL}_2^3/\widetilde{N}$, where

$$\widetilde{N} = \{(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_i = \pm 1 \text{ for } i = 1,2,3 \text{ and } \varepsilon_1\varepsilon_2\varepsilon_3 = 1\}.$$

The Mumford–Tate group of the M-T unliftable and M-T decomposed weak M-T lift B of A satisfies $G^{\mathrm{der}}_{B/\overline{\mathbf{Q}}} \cong \mathrm{SL}_2^3$. This is the example studied in detail in [Noo01].

2.14.2 There exist simple abelian varieties $A/\mathbf{C}$ for which G_A^{ad} is not simple. One can construct such an example where $G_A^{\mathrm{der}} \cong G_1 \times G_1/N$ with

$$G_{1/\overline{\mathbf{Q}}} \cong G_{2/\overline{\mathbf{Q}}} \cong \mathrm{SL}_2^2$$

and $N_{/\overline{\mathbf{Q}}} = \{(\pm 1, \pm 1)\} \subset \mathrm{SL}_2^2 \cong G_i$ embedded diagonally into $G_1 \times G_2$. In this case, the M-T unliftable and M-T decomposed weak M-T lift B of A is a product $B = B_1 \times B_2$ and $G^{\mathrm{der}}_{B_i} = G_i$.

2.14.3 There exist simple abelian varieties $A/\mathbf{C}$ for which G_A^{ad} is absolutely simple of type $D_k^{\mathbf{R}}$, with k even, $\mathscr{G}_{\mathbf{Q}}$ acting trivially on the Dynkin diagram and where $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ decomposes over $\overline{\mathbf{Q}}$ as a multiple of the representation of G_A^{der} with highest weight ϖ_k. Since G_A^{der} acts faithfully on V, it is not simply connected. Let B be the M-T unliftable and M-T decomposed weak M-T lift of A. Then G_B^{der} is the universal cover of G_A^{der} and B is a product $B \cong A \times B'$, where $W = \mathrm{H}^1_{\mathrm{B}}(B'(\mathbf{C}), \mathbf{Q})$ decomposes over $\overline{\mathbf{Q}}$ as a multiple of the representation of G_B^{der} with highest weight ϖ_{k-1}.

2.15 Corollary. *For every abelian motive $M/\mathbf{C}$ there exists an essentially Mumford–Tate unliftable and Mumford–Tate decomposed abelian variety $B/\mathbf{C}$ which provides a weak Mumford–Tate lift for M.*

Proof. There exist an abelian variety $A/\mathbf{C}$ and a surjection of the corresponding Mumford–Tate groups $G_A \to G_M$ commuting with the maps h_A and h_M. Let $B'/\mathbf{C}$ be the essentially Mumford–Tate unliftable and Mumford–Tate decomposed weak Mumford–Tate lift for A provided by theorem 2.12. This gives a morphism $\pi_{B'}\colon G_{B'}^{\mathrm{der}} \to G_M^{\mathrm{der}}$. The fact that B' is Mumford–Tate decomposed implies that there is an isogeny $B' \sim \prod_{i\in I} B_i$, where the B_i are M-T decomposed abelian varieties such that the groups $G_{B_i}^{\mathrm{der}}$ are $\mathbf{Q}$-simple, and such that $G_{B'}^{\mathrm{der}} \cong \prod_{i\in I} G_{B_i}^{\mathrm{der}}$. Let $J \subset I$ be the subset of indices i such that $G_{B_i}^{\mathrm{der}}$ is not in the kernel of $\pi_{B'}$ and let $B = \prod_{i\in J} B_i$. Then $G_B = \prod_{i\in J} G_{B_i}$ and $\left(\pi_{B'}\right)_{|G_B^{\mathrm{der}}}$ is an isogeny from G_B^{der} onto G_M^{der}, so B verifies the condition of the corollary. □

3 Mumford–Tate liftings and motives

3.1 Proposition. *Suppose that $A_{/\mathbf{C}}$ and $B_{/\mathbf{C}}$ are abelian varieties over $\mathbf{C}$, let (G_A, h_A) and (G_B, h_B) be the associated Mumford–Tate data and assume that there exists an isomorphism $G_A^{\mathrm{ad}} \cong G_B^{\mathrm{ad}}$ such that $\pi_A^{\mathrm{ad}} \circ h_A = \pi_B^{\mathrm{ad}} \circ h_B$.*

Let $F \subset \mathbf{C}$ be an algebraically closed field. Then there exists an abelian variety A/F such $A \otimes_F \mathbf{C} = A_{/\mathbf{C}}$ if and only if there exists an abelian variety B/F such that $B \otimes_F \mathbf{C} = B_{/\mathbf{C}}$.

Proof. This is proved as in [Noo01, 4.5]. Let $h_A\colon S \to G_{A/\mathbf{R}}$ and $h_B\colon S \to G_{B/\mathbf{R}}$ be the maps defining the Hodge structures on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q})$ respectively. Let X_A and X_B be the $G_A(\mathbf{R})$- and $G_B(\mathbf{R})$-conjugacy classes of h_A and h_B. The main theorem of [Del79] implies that for all compact open subgroups $K_A \subset G_A(\mathbf{A}_f)$ and $K_B \subset G_B(\mathbf{A}_f)$, one can construct quasi-canonical models ${}_{K_A}M_{/\overline{\mathbf{Q}}}(G_A, X_A)^0$ and ${}_{K_B}M_{/\overline{\mathbf{Q}}}(G_B, X_B)^0$ over $\overline{\mathbf{Q}}$ of the corresponding connected Shimura varieties. Here $\overline{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$.

For K_A and K_B sufficiently small, there exist “universal” abelian schemes $\mathscr{A} \to {}_{K_A}M_{/\overline{\mathbf{Q}}}(G_A, X_A)^0$ and $\mathscr{B} \to {}_{K_B}M_{/\overline{\mathbf{Q}}}(G_B, X_B)^0$ and points

$$a \in {}_{K_A}M_{/\overline{\mathbf{Q}}}(G_A, X_A)^0(\mathbf{C}),\ b \in {}_{K_B}M_{/\overline{\mathbf{Q}}}(G_B, X_B)^0$$

such that $A = \mathscr{A}_a$ and $b = \mathscr{B}_b$.

Let $G^{\mathrm{ad}} = G_A^{\mathrm{ad}} \cong G_B^{\mathrm{ad}}$. One can choose K_A, K_B and $K^{\mathrm{ad}} \subset G^{\mathrm{ad}}(\mathbf{A}_f)$ such that, in addition to the above conditions, there is a diagram

$$\begin{tikzcd} {}_{K_A}M_{/\overline{\mathbf{Q}}}(G_A, X_A)^0 \arrow[dr] & & {}_{K_B}M_{/\overline{\mathbf{Q}}}(G_B, X_B)^0 \arrow[dl] \\ & {}_{K^{\mathrm{ad}}}M_{/\overline{\mathbf{Q}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^0, & \end{tikzcd}$$

in which each arrow is a quotient map for the action of a finite group and hence is a finite morphism (cf. [Del79, 2.7.11(b)]) and such that a and b have the same image in ${}_{K^{\mathrm{ad}}}M_{/\overline{\mathbf{Q}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})(\mathbf{C})^0$. The proposition follows. ☐

3.2 Corollary. *The statement of the proposition is true in particular if $B_{/\mathbf{C}}$ provides a (weak) Mumford–Tate lift of $A_{/\mathbf{C}}$.*

3.3 Proposition. *Let $F \subset \mathbf{C}$ be an algebraically closed field and let A and B abelian varieties over F such that $B_{/\mathbf{C}}$ provides a Mumford–Tate lift of $A_{/\mathbf{C}}$. Then the motive $h^1(A)$ belongs to the category $\langle h^1(B), \mathbf{Q}(1)\rangle$.*

The map $G_B \to G_A$ induced by the Betti realization of this inclusion is the map given by the structure of B as a Mumford–Tate lift of A.

Proof. This generalises [Noo01, 4.8–4.11].

As explained in 1.5, it follows from [DM82, 6.25] that the Betti realization induces an equivalence of $\langle h^1(B), \mathbf{Q}(1)\rangle$ with $\langle \mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q}), \mathbf{Q}(1)\rangle$, the tannakian subcategory of the category of Hodge structures generated by $\mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q})$ and the Tate Hodge structure $\mathbf{Q}(1)$. To prove the proposition, it therefore suffices to show that the Hodge structure $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ belongs to $\langle \mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q}), \mathbf{Q}(1)\rangle$. For the rest of the proof, we write $V_A = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and $V_B = \mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q})$.

By definition of the Mumford–Tate group, the underlying vector space of any object of the category $\langle V_B, \mathbf{Q}(1)\rangle$ of Hodge structures naturally carries the structure of a representation of G_B. This gives a $\otimes$-equivalence of $\langle V_B, \mathbf{Q}(1)\rangle$ with the subcategory $\langle V_B, \mathbf{Q}(1)\rangle_{\mathbf{Rep}}$ of $\mathbf{Rep}_{\mathbf{Q}}(G_B)$. As we saw in 1.5, for any object of W the latter category, the Hodge structure is given by composing the morphism $h_B \colon S \to G_{B/\mathbf{R}}$ with the action of G_B on W.

Let (G_A, h_A) be the Mumford–Tate datum associated to A. By hypothesis, there exists a central morphism $\pi\colon G_B \to G_A$ such that $h_A = \pi_{\mathbf{R}} \circ h_B$. This makes every $\mathbf{Q}$-linear representation W of G_A into a representation of G_B and,

for any such W, it carries the Hodge structure on W defined by h_A into the Hodge structure defined by h_B. To prove the proposition it is therefore sufficient to show that V_A, considered as representation of G_B, belongs to $\langle V_B, \mathbf{Q}(1)\rangle_{\mathbf{Rep}}$. This follows immediately, because V_B is a faithful representation of G_B and $V_B \otimes \mathbf{Q}(1)$ is its dual, so one has

$$\langle V_B, \mathbf{Q}(1)\rangle_{\mathbf{Rep}} = \mathbf{Rep}_{\mathbf{Q}}(G_B),$$

cf. [DM82, 2.20] and its proof. □

3.4 Proposition. *Let A and B be abelian varieties over an algebraically closed field $F \subset \mathbf{C}$ such that $B_{/\mathbf{C}}$ provides a weak Mumford–Tate lift of $A_{/\mathbf{C}}$. Then $h^1(A)$ belongs to $\langle h^1(B), (\mathbf{CM})_F\rangle$.*

Taking the Betti realization, this inclusion induces a map between the corresponding Mumford–Tate groups. On the derived group, this map is the map $\pi^{\mathrm{der}}\colon G_B^{\mathrm{der}} \to G_A^{\mathrm{der}}$ given by the structure of B as weak Mumford–Tate lift of A.

Proof. Let T_A and T_B be the connected components of the centres of G_A and of G_B respectively. We write $V_A = \mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and $V_B = \mathrm{H}^1_{\mathrm{B}}(B(\mathbf{C}), \mathbf{Q})$ as in the proof of 3.3 and consider all spaces we encounter as representations of $H = T_A \times T_B \times G_B^{\mathrm{der}}$. In the case of V_A, the group H acts via

$$H \longrightarrow T_A \times G_B^{\mathrm{der}} \xrightarrow{\mathrm{id}\times\pi^{\mathrm{der}}} T_A \times G_A^{\mathrm{der}} \longrightarrow G_A$$

and in the case of V_B it acts via $H \to T_B \times G_B^{\mathrm{der}} \to G_B$.

The groups T_A and T_B act on V_A and V_B. Composing these representations with the projections $H \to T_A$ and $H \to T_B$ respectively we obtain representations V_A^c and V_B^c of H. As $W = V_B \oplus V_A^c \oplus V_B^c$ is a faithful representation of H, it follows that V_A belongs to the subcategory of $\mathbf{Rep}_{\mathbf{Q}}(H)$ $\otimes$-generated by W and its dual. As there is an isomorphism of Hodge structures $V_B^\vee \cong V_B \otimes \mathbf{Q}(1)$, this in turn implies that V_A belongs to the tannakian subcategory of $\mathbf{Rep}_{\mathbf{Q}}(H)$ generated by V_B and the abelian representations of H, i. e. the representations where H acts through a commutative quotient.

We fix an irreducible direct factor W_2 of V_A in $\mathbf{Rep}_{\mathbf{Q}}(H)$. There is an irreducible abelian $\mathbf{Q}$-linear representation W_3 of H such that W_2 is isomorphic to a subobject of $W_3 \otimes V_B^{\otimes d}$. Replace $V_B^{\otimes d}$ by an irreducible direct factor W_1 such that W_2 still is a subobject of $W_3 \otimes W_1$. The projection of $W_1 \otimes W_1^\vee$ onto the trivial

representation induces a surjection $W_3 \otimes W_1 \otimes W_1^\vee \twoheadrightarrow W_3$. The composite of this surjection with the map

$$W_2 \otimes W_1^\vee \hookrightarrow W_3 \otimes W_1 \otimes W_1^\vee$$

deduced from the inclusion $W_2 \hookrightarrow W_3 \otimes W_1$ is a map $W_2 \otimes W_1^\vee \to W_3$. It is not difficult to check that this map is non-trivial and as W_3 was assumed to be irreducible, this implies that the map is surjective. This proves that W_3 belongs to the tannakian subcategory of $\mathbf{Rep}_{\mathbf{Q}}(H)$ which is $\otimes$-generated by $W_1^\vee$ and W_2.

It follows that the action of H on W_3 factors through $G_A \times G_B$ and as W_3 is an abelian representation of H, it is also an abelian representation of $G_A \times G_B$. This means that W_3 carries a Hodge structure of CM-type and hence that V_A belongs to $\langle V_B, (\mathbf{CM})\text{-}\mathbf{Hodge}\rangle$ as required. $\square$

3.5 Corollary. *Let A and B be abelian varieties over F and let G_A and G_B be their respective Mumford–Tate groups. Assume that B provides a weak Mumford–Tate lift of A and that $G_A^{\mathrm{der}} \cong G_B^{\mathrm{der}}$. Then the categories $\langle h^1(A), (\mathbf{CM})_F\rangle$ and $\langle h^1(B), (\mathbf{CM})_F\rangle$ coincide.*

3.6 Corollary. *Let $F \subset \mathbf{C}$ be an algebraically closed field and let M be an object of $(\mathbf{AV})_F$. Then there exist an essentially Mumford–Tate unliftable and Mumford–Tate decomposed abelian variety A over F such that M belongs to*

$$\langle h^1(A), (\mathbf{CM})_F\rangle.$$

4 Motivic liftings of Galois representations

4.1 Assume that $F \subset \mathbf{C}$ is a number field, M an abelian motive over F and (G_M, h_M) the Mumford–Tate datum associated to M. As we recalled in 1.6, the p-adic Galois representation associated to the étale realization of M factors through a morphism

$$\rho_{M,p} \colon \mathscr{G}_F \longrightarrow G_M(\mathbf{Q}_p).$$

There exists a finite extension $F' \supset F$ such that the Mumford–Tate group of $M_{F'}$ is connected. The Mumford–Tate group of $M_{F'}$ is then equal to the connected component of G_M. In what follows, we will assume that G_M is already connected (replacing F by a finite extension if necessary).

It follows from theorems of Tsuji, [Tsu99] and De Jong, [dJ96] that, for every p, the representation $\rho_{M,p}$ of $\mathscr{G}_F$ on $\mathrm{H}_{\text{ét}}(M, \mathbf{Q}_p)$ is geometric in the sense of Fontaine and Mazur, [FM95, §1]. Here a representation of $\mathscr{G}_F$ on a finite dimensional $\mathbf{Q}_p$-vector space is called *geometric* if

- it is unramified outside a finite set of non-archimedean places of F and
- for each valuation $\bar{v}$ of $\bar{F}$ of residue characteristic p, the restriction to the inertia group $\mathscr{I}_{F,\bar{v}}$ is potentially semi-stable (cf. [Fon94]).

More generally, for a linear algebraic group G over $\mathbf{Q}_p$ and a continuous morphism $\rho_p \colon \mathscr{G}_F \longrightarrow G(\mathbf{Q}_p)$, we will say that ρ_p is *geometric* if there exists a faithful representation V_p of G such that the resulting representation of $\mathscr{G}_F$ on V_p is geometric. This is the case if and only if the representation of $\mathscr{G}_F$ on W_p is geometric for any representation W_p of G.

4.2 Definition. Assume that G and $\widetilde{G}$ are linear algebraic groups over $\mathbf{Q}$ or over $\mathbf{Q}_p$ and that $\rho_p \colon \mathscr{G}_F \to G(\mathbf{Q}_p)$ and $\tilde{\rho}_p \colon \mathscr{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ are geometric Galois representations. We will say that $\tilde{\rho}_p$ is a *geometric lift* of ρ_p if there exists a central isogeny $\pi \colon \widetilde{G} \to G$ such that $\rho_p = \pi \circ \tilde{\rho}_p$. If ρ_p does not admit any geometric lift with $\ker \pi$ non-trivial, it will be called *geometrically unliftable*.

If there exists a central isogeny $\pi^{\mathrm{der}} \colon \widetilde{G}^{\mathrm{der}} \to G^{\mathrm{der}}$ such that there is an equality $\tilde{\pi}^{\mathrm{ad}} \circ \tilde{\rho}_p = \pi^{\mathrm{ad}} \circ \rho_p$ then $\tilde{\rho}_p$ is said to be a *weak geometric lift* of ρ_p. As in 2.1, the maps π^{ad} and $\tilde{\pi}^{\mathrm{ad}}$ are the projections $G \to G^{\mathrm{ad}}$ and $\widetilde{G} \to \widetilde{G}^{\mathrm{ad}}$. The central isogeny π^{der} induces an isomorphism $G^{\mathrm{ad}} \cong \widetilde{G}^{\mathrm{ad}}$, giving a sense to the equality $\tilde{\pi}^{\mathrm{ad}} \circ \tilde{\rho}_p = \pi^{\mathrm{ad}} \circ \rho_p$. We will say that ρ_p is *essentially geometrically unliftable* if it does not admit a weak geometric lift with $\ker \pi^{\mathrm{der}}$ non-trivial.

4.3 Proposition. *Suppose that $F \subset \mathbf{C}$ is a number field, A/F an abelian variety with connected Mumford–Tate group G_A and that B is an abelian variety over a finite extension $F' \supset F$ such that $B_{/\mathbf{C}}$ provides a Mumford–Tate lift of $A_{/\mathbf{C}}$.*

The number field F' can be chosen such that, for every prime number p, the Galois representation $\rho_{A,p}$ of $\mathscr{G}_{F'}$ on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ belongs to the subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{F'})$ $\otimes$-generated by $\mathrm{H}^1_{\text{ét}}(B_{\bar{F}}, \mathbf{Q}_p)$ and $\mathbf{Q}_p(1)$. For every p, the morphism $G_B \to G_A$ then realises the Galois representation $\rho_{B,p}$ as a geometric lift of $\rho_{A,p}$.

Proof. By proposition 3.3, the motive $h^1(A_{\bar{F}})$ belongs to $\langle h^1(B_{\bar{F}}, \mathbf{Q}(1)\rangle$ and, for F' large enough, this is already the case over F'. Taking the p-adic étale re-

alizations, this implies the corresponding statement for the Galois representations. The inclusion of $h^1(A_{F'})$ in $\langle h^1(B_{F'}), \mathbf{Q}(1)\rangle$ corresponds to a morphism $G_B \to G_A$ and taking the p-adic étale realizations this gives rise to a commutative diagram

$$\begin{tikzcd} & G_B(\mathbf{Q}_p) \arrow[d] \\ \mathscr{G}_{F'} \arrow[ur, "\rho_{B,p}"] \arrow[r, "\rho_{A,p}"] & G_A(\mathbf{Q}_p) \end{tikzcd}$$

proving the proposition. □

4.4 As above, assume that F is a number field contained in $\mathbf{C}$. Following the notations of [FM95, §6], let $(\mathbf{CM})$-$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$, or $(\mathbf{CM})$-$\mathbf{Rep}$ if no confusion is likely, denote the tannakian subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ consisting of the potentially abelian geometric representations, in other words, the geometric representations such that the restriction to a subgroup of $\mathscr{G}_F$ of finite index has abelian image. It follows from [FM95, §6] that $(\mathbf{CM})$-$\mathbf{Rep}$ is the tannakian subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ generated by the representations factoring through finite groups and the representations of the form $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ for A/F an abelian variety which is potentially of CM-type. Thus, $(\mathbf{CM})$-$\mathbf{Rep}$ is the tannakian category of the p-adic étale realizations of the objects of $(\mathbf{CM})_F$.

4.5 Proposition. *Suppose that $F \subset \mathbf{C}$ is a number field, A/F an abelian variety with connected Mumford–Tate group G_A and that B is an abelian variety over a finite extension $F' \supset F$ such that the Mumford–Tate group G_B is connected and $B_{/\mathbf{C}}$ provides a weak Mumford–Tate lift of $A_{/\mathbf{C}}$.*

For every prime number p, the representation $\rho_{A,p}$ of $\mathscr{G}_{F'}$ on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ is an object of the subcategory $\langle \mathrm{H}^1_{\text{ét}}(B_{\bar{F}}, \mathbf{Q}_p), (\mathbf{CM})\text{-}\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{F'})\rangle$ of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{F'})$. Via the map $G_B^{\mathrm{der}} \to G_A^{\mathrm{der}}$, the Galois representation $\rho_{B,p}$ provides a weak geometric lift of $(\rho_{A,p})_{|\mathscr{G}_{F'}}$.

Proof. It follows from proposition 3.4 that there is a finite extension F'' of F' such that the motive $h^1(A_{F''})$ belongs to $\langle h^1(B_{F''}), (\mathbf{CM})_{F''}\rangle$. As the Mumford–Tate group of $\langle h^1(B_{F''}), (\mathbf{CM})_{F''}\rangle$ is a pro-algebraic group with derived group G_B^{der}, this gives rise to an isogeny $\pi\colon G_B^{\mathrm{der}} \to G_A^{\mathrm{der}}$ and π induces an isomorphism $\pi^{\mathrm{ad}}\colon G_B^{\mathrm{ad}} \to G_A^{\mathrm{ad}}$. Taking p-adic étale realizations, there is a commutative dia-

gram

$$\begin{tikzcd} & G_B(\mathbf{Q}_p) \arrow[dr] & \\ \mathscr{G}_{F''} \arrow[ur, "\rho_{B,p}"] \arrow[dr, "\rho_{A,p}"'] & & G_A^{\mathrm{ad}}(\mathbf{Q}_p) = G_B^{\mathrm{ad}}(\mathbf{Q}_p). \\ & G_A(\mathbf{Q}_p) \arrow[ur] & \end{tikzcd} \tag{4.5.$*$}$$

This proves all statements of the proposition with F'' instead of F'.

Faltings' theorem, [Fal83, Satz 4] implies that if C is the commuting algebra of G_A in $\mathrm{End}(\mathrm{H}^1_B(A(\mathbf{C}), \mathbf{Q}))$, then $C \otimes \mathbf{Q}_p$ is the commuting algebra of $\rho_{A,p}(\mathscr{G}_{F''})$ in $\mathrm{End}(\mathrm{H}^1_{\text{ét}}(A(\mathbf{C}), \mathbf{Q}_p))$. This implies that that the centralizer of the image of $\mathscr{G}_{F''}$ in $G_A^{\mathrm{ad}}(\mathbf{Q}_p)$ is trivial. It follows from lemma 4.6 that the diagram 4.5.$*$ also commutes with $\mathscr{G}_{F''}$ replaced by $\mathscr{G}_{F'}$. Lemma 4.7 implies that the representation of $\mathscr{G}_{F'}$ on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$ is an object of $\langle \mathrm{H}^1_{\text{ét}}(B_{\bar{F}}, \mathbf{Q}_p), (\mathbf{CM})\text{-}\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{F'})\rangle$. $\square$

4.6 Lemma. *Assume that p is a prime number and that G is a connected linear algebraic group over $\mathbf{Q}_p$. Let F be a number field and let $\rho_1, \rho_2 \colon \mathscr{G}_F \to G(\mathbf{Q}_p)$ be Galois representations coinciding on $\mathscr{G}_{F'}$ for some finite extension F' of F. Also assume that the centralizer of $\rho_1(\mathscr{G}_{F'}) = \rho_2(\mathscr{G}_{F'})$ in G is trivial. Then $\rho_1 = \rho_2$ on $\mathscr{G}_F$.*

Proof. It is sufficient to treat the case where F' is a Galois extension of F.

Let $\delta \colon \mathscr{G}_F \to G(\mathbf{Q}_p)$ be defined by $\delta(\sigma) = \rho_1(\sigma)\rho_2(\sigma)^{-1}$. This map satisfies the cocycle condition $\delta(\sigma\tau) = \delta(\sigma)\left(\rho_2(\sigma)\delta(\tau)\rho_2(\sigma)^{-1}\right)$. This implies that δ is constant on the classes $\sigma\mathscr{G}_{F'}$ for $\sigma \in \mathscr{G}_F$. As $\mathscr{G}_{F'}$ is normal, it follows that δ is also constant on the classes $\mathscr{G}_{F'}\sigma$ and it follows that for all $\sigma \in \mathscr{G}_F$ and $\tau \in \mathscr{G}_{F'}$ one has $\delta(\sigma) = \rho_2(\tau)\delta(\sigma)\rho_2(\tau)^{-1}$. Therefore $\delta(\sigma)$ lies in the centralizer of $\rho_2(\mathscr{G}_{F'})$ and we conclude that δ is trivial. $\square$

4.7 Lemma. *Let p be a prime number, G_1 and G_2 connected linear algebraic groups over $\mathbf{Q}_p$ and $\pi^{\mathrm{der}} \colon G_1^{\mathrm{der}} \to G_2^{\mathrm{der}}$ a central isogeny. Let V_1 be a faithful $\mathbf{Q}_p$-linear representation of G_1 and let V_2 be any $\mathbf{Q}_p$-linear representation of G_2.*

Let F be a number field and let $\rho_i \colon \mathscr{G}_F \to G_i(\mathbf{Q}_p)$ (for $i = 1, 2$) be geometric Galois representations. Assume that $V_1^\vee$ lies in the subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ $\otimes$-generated by V_1 and $(\mathbf{CM})$-$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ and that $\pi_1^{\mathrm{ad}} \circ \rho_1 = \pi_2^{\mathrm{ad}} \circ \rho_2$, where the $\pi_i^{\mathrm{ad}} \colon G_i \to G_i^{\mathrm{ad}}$ are the canonical projections and $G_1^{\mathrm{ad}} = G_2^{\mathrm{ad}}$ is the identification induced by π^{der}. Then V_2 is an object of the subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ $\otimes$-generated by V_1 and $(\mathbf{CM})$-$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$.

Proof. Fix an object V_3 of $(\mathbf{CM})$-$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ such that $V_1^\vee$ lies in the subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ $\otimes$-generated by V_1 and V_3. Let T_3 be the Zariski closure of the image of $\mathscr{G}_F$ in $\mathrm{GL}(V_3)$ and let $\rho_3 \colon \mathscr{G}_F \to T_3(\mathbf{Q}_p)$ be the morphism giving the action of $\mathscr{G}_F$ on V_3.

Let T_1 and T_2 be the connected components of the centres of G_1 and of G_2 respectively. Put $H = T_1 \times T_2 \times T_3 \times G_1^{\mathrm{der}}$ and consider the natural maps $H \to G_1$ and $H \to G_2$. Via these maps, we consider V_1 and V_2 as representations of H. It can be shown exactly as in the proof of proposition 3.4 that V_2 belongs to the subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(T_3 \times G_A \times G_B)$ generated by V_1 and the abelian representations. For any abelian representation W_3 of $T_3 \times G_A \times G_B$, the induced Galois representation belongs to $(\mathbf{CM})$-$\mathbf{Rep}(\mathscr{G}_F)$ so this implies that V_2 belongs to the subcategory $\langle V_1, (\mathbf{CM})\text{-}\mathbf{Rep}(\mathscr{G}_F)\rangle$ of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$. □

4.8 We keep the above notations, i. e. $F \subset \mathbf{C}$ is a number field, A/F an abelian variety and G_A its Mumford–Tate group, which is assumed to be connected. For each prime number p, we denote by $\rho_{A,p} \colon \mathscr{G}_F \to G_A(\mathbf{Q}_p)$ the p-adic Galois representation associated to A.

Fix an algebraic group $\widetilde{G}$ over $\mathbf{Q}$ and a central isogeny $\widetilde{G}^{\mathrm{der}} \to G_A^{\mathrm{der}}$ and suppose that for every p in a set P of prime numbers $\tilde{\rho}_p \colon \mathscr{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ is a weak geometric lift of $\rho_{A,p}$.

4.9 Theorem. *Assume that any simple factor of $\widetilde{G}^{\mathrm{der}}_{/\mathbf{C}}$ lying over a factor of $G^{\mathrm{der}}_{A/\mathbf{C}}$ of type $D_k^{\mathbf{H}}$ is a quotient of the h-maximal cover, cf. 2.5. Let F' be a finite extension of F and B an essentially Mumford–Tate unliftable abelian variety over F' with connected Mumford–Tate group G_B such that $B_{/\mathbf{C}}$ provides a weak Mumford–Tate lift of $A_{/\mathbf{C}}$. For each prime number p, let $V_{B,p} = \mathrm{H}^1_{\text{ét}}(B_{\bar{F}}, \mathbf{Q}_p)$ be the p-adic representation of $\mathscr{G}_{F'}$ associated to B.*

Then the map $G_B^{\mathrm{der}} \to G_A^{\mathrm{der}}$ lifts to $G_B^{\mathrm{der}} \to \widetilde{G}^{\mathrm{der}}$. For every $p \in P$

- *the representation $\rho_{B,p}$ is a weak geometric lift of the restriction $\left(\tilde{\rho}_p\right)_{|\mathscr{G}_{F'}}$ and*
- *for every representation $\widetilde{V}_p$ of $\widetilde{G}_{/\mathbf{Q}_p}$, the Galois representation on $\widetilde{V}_p$ is an object of $\langle V_{B,p}, (\mathbf{CM})\text{-}\mathbf{Rep}\rangle$.*

Proof. We fix a faithful self-dual representation $\widetilde{V}$ of $\widetilde{G}$ and for each prime number p we write $\widetilde{V}_p = \widetilde{V} \otimes_{\mathbf{Q}} \mathbf{Q}_p$. As $\widetilde{V}$ generates the tannakian category of representations of $\widetilde{G}$, it is sufficient to prove the corollary for the representations $\widetilde{V}_p$.

Since B is essentially Mumford–Tate unliftable, it follows from theorem 2.9 that G_B^{der} is h-maximal. It follows that the map $G_B^{\mathrm{der}} \to G_A^{\mathrm{der}}$ lifts to a map $G_B^{\mathrm{der}} \to \widetilde{G}^{\mathrm{der}}$. Let $\rho_{B,p} \colon \mathscr{G}_{F'} \to G_B(\mathbf{Q}_p)$ be the map giving the Galois representation on $V_{B,p}$. Write $\pi_A^{\mathrm{ad}} \colon G_A \to G_A^{\mathrm{ad}}$, $\pi_B^{\mathrm{ad}} \colon G_B \to G_B^{\mathrm{ad}}$ and $\tilde{\pi}^{\mathrm{ad}} \colon \widetilde{G} \to \widetilde{G}^{\mathrm{ad}}$ for the projections. For any $p \in P$, the proposition 4.5 and the fact that $\tilde{\rho}_p$ is a weak geometric lift of $\rho_{A,p}$ imply that $\tilde{\pi}^{\mathrm{ad}} \circ \tilde{\rho}_{p|\mathscr{G}_{F'}} = \pi_A^{\mathrm{ad}} \circ \rho_{A,p|\mathscr{G}_{F'}} = \pi_B^{\mathrm{ad}} \circ \rho_{B,p}$. The remaining statement of the theorem follows from lemma 4.7. $\square$

4.10 Important remark. Concerning the abelian variety B/F' which appears in propositions 4.3 and 4.5 and in theorem 4.9, it follows from theorem 2.12 and proposition 3.1 that there exist a number field F' and an essentially M-T unliftable and M-T decomposed weak Mumford–Tate lift B/F' of A as in the propositions and in the theorem. The condition that G_B is connected can be forced by replacing F' by a finite extension.

4.11 Corollary. *Let notations be as in 4.8 with $\widetilde{G}^{\mathrm{der}} \to G_A^{\mathrm{der}}$ satisfying the hypotheses of the theorem. Then, for every $p \in P$ and every representation $\widetilde{V}_p$ of $\widetilde{G}_{/\mathbf{Q}_p}$, the induced representation of $\mathscr{G}_F$ on $\widetilde{V}_p$ occurs in the p-adic étale realization of an abelian motive.*

Proof. The remark and the theorem imply that there is is a finite extension F' of F such that the representation of $\mathscr{G}_{F'}$ on $\widetilde{V}_p$ occurs in the p-adic étale realization of an object M' of $(\mathbf{AV})_{F'}$. The representation of $\mathscr{G}_F$ on $\widetilde{V}_p$ then occurs in the p-adic étale realization of the Weil restriction $\mathrm{Res}_{F'/F} M'$, which is also in $(\mathbf{AV})_F$. $\square$

5 Abelian varieties with Mumford–Tate group of type $D_k^{\mathbf{H}}$

5.1 The following notation and hypotheses will be in force until definition 5.10. We let $A/\mathbf{C}$ be a simple abelian variety and (G_A, h_A) the associated Mumford–Tate datum. This implies that G_A is connected. We assume throughout that A is essentially Mumford–Tate unliftable, Mumford–Tate decomposed and that G_A is of type $D_k^{\mathbf{H}}$ with $k \geq 4$.

It follows that there exist a totally real number field K_0 and an absolutely simple algebraic group G^s/K_0 such that $G_A^{\mathrm{der}} = \mathrm{Res}_{K_0/\mathbf{Q}} G^s$. By assumption, the group G^s/K_0 is h-maximal in the sense of 2.5. The representation of G_A^{der} on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ decomposes over $\overline{\mathbf{Q}}$ as a multiple of the direct sum of the standard (orthogonal) representations of the different factors of $G^{\mathrm{der}}_{A/\overline{\mathbf{Q}}}$. As in [Del79]

and in §2 of this paper, denote by $I = \{\iota\colon K_0 \hookrightarrow \mathbf{C}\}$ the set of complex embeddings of K_0. As K_0 is totally real, I is also the set of real embeddings of K_0. The Dynkin diagram of $G^{\mathrm{der}}_{A/\mathbf{C}}$ is a disjoint union, indexed by I, of diagrams of type D_k. The Hodge cocharacter $\mu_A\colon \mathbf{G}_{m/\mathbf{C}} \to G_{/\mathbf{C}}$ associated to A projects trivially on some factors of $G^{\mathrm{ad}}_{/\mathbf{C}}$. On the other factors, the conjugacy class of the projection is dual to one of the vertices α_{k-1} or α_k (or possibly α_1 if $k = 4$) of the corresponding component of the Dynkin diagram. Without loss of generality, we will henceforth assume that, on these factors, it corresponds to α_k. As in section 2, let $I_c \subset I$ be the set of embeddings corresponding to the factors onto which μ_A projects trivially and let $I_{nc} = I - I_c$. Recall that I_c is the set of embeddings $\iota\colon K_0 \hookrightarrow \mathbf{R}$ such that the factor of $G^{\mathrm{der}}_{A/\mathbf{R}}$ corresponding to ι is compact and I_{nc} is the set of real embeddings of K_0 for which the corresponding factor of $G^{\mathrm{der}}_{A/\mathbf{R}}$ is non-compact.

As the Hodge filtration and its complex conjugate are opposite filtrations, the complex conjugate of μ is conjugate to μ^{-1}, up to a central cocharacter. This implies that complex conjugation acts on the Dynkin diagram by the main involution and hence that it acts trivially if k is even and exchanges α_{k-1} and α_k on every factor if k is odd. It follows that $\mathscr{G}_{\mathbf{Q}}$ (or $\mathrm{Aut}(\mathbf{C})$) acts on the Dynkin diagram through $\mathrm{Gal}(K/\mathbf{Q})$ for a number field $K \supset K_0$ with $[K : K_0] = 1$ or 2 and which is totally real if k is even, a CM field if k is odd. In particular $[K : K_0] = 2$ if k is odd. Note that the statement is also true for $k = 4$, because it follows from the definition of the case $D_4^{\mathbf{H}}$ (see 2.5) that the vertices α_1 of the connected components of the Dynkin diagram form a $\mathscr{G}_{\mathbf{Q}}$-orbit, so the stabilizer of a connected component is of order at most 2.

5.2 Construction of $\widetilde{G}$ and $\tilde{\mu}$. We aim to construct an algebraic group $\widetilde{G}/\mathbf{Q}$ such that $\widetilde{G}^{\mathrm{der}}$ is simply connected and that $\widetilde{G}^{\mathrm{ad}} = G^{\mathrm{ad}}_A$, together with a cocharacter $\tilde{\mu}$ of $\widetilde{G}_{/\mathbf{C}}$ such that $\tilde{\pi}^{\mathrm{ad}} \circ \tilde{\mu} = \pi^{\mathrm{ad}} \circ \mu$. The argument is strongly inspired by [Del79], see also the proof of theorems 2.9 and 2.12. Here, as before, π^{ad} and $\tilde{\pi}^{\mathrm{ad}}$ are the projections $G_A \to G^{\mathrm{ad}}_A$ and $\widetilde{G} \to G^{\mathrm{ad}}_A$ respectively.

Let $\widetilde{G}^s/K_0$ be the simply connected cover of G^s. Consider the direct sum of the representations of $\widetilde{G}^s_{/\overline{\mathbf{Q}}}$ with highest weights ϖ_{k-1} and ϖ_k. A multiple of that representation can be defined over K_0, let W^s be the resulting representation of $\widetilde{G}^s$. By construction, there is a decomposition $W^s \otimes_{K_0} K = W^s_1 \oplus W^s_2$, where W^s_1 (resp. W^s_2) is a multiple of the representation with highest weight

ϖ_{k-1} (resp. ϖ_k). A non-trivial element of $\mathrm{Gal}(K/K_0)$ exchanges the factors W_1^s and W_2^s of this decomposition. If $[K : K_0] = 2$, then the composite map $W^s \subset W^s \otimes_{K_0} K \twoheadrightarrow W_1^s$ is an isomorphism of K_0-vector spaces and endows W^s with a structure of K-vector space.

Put $\widetilde{G}^{\mathrm{der}} = \mathrm{Res}_{K_0/\mathbf{Q}} \widetilde{G}^s$ and let W be the rational representation of $\widetilde{G}^{\mathrm{der}}$ deduced from W^s. Since W is the underlying $\mathbf{Q}$-vector space of W^s, it carries a structure of K-vector space. The cocharacter $\pi^{\mathrm{ad}} \circ \mu$ of $G^{\mathrm{ad}}_{/\mathbf{C}}$ lifts to a quasi-cocharacter ν of $\widetilde{G}^{\mathrm{der}}_{/\mathbf{C}}$, cf. 1.1.

5.3 Lemma. *There is a decomposition*

$$W \otimes_{\mathbf{Q}} \mathbf{C} \cong \bigoplus_{\iota \in I} \left(W_1^{(\iota)} \oplus W_2^{(\iota)} \right) \tag{5.3.$*$}$$

such that, for each $\iota \in I$, the representation $W_1^{(\iota)}$ (resp. $W_2^{(\iota)}$) is a multiple of the irreducible representation with highest weight ϖ_{k-1} (resp. ϖ_k) of the factor of $G^{\mathrm{der}}_{A/\mathbf{C}}$ corresponding to ι.

The weights of ν on the $W_j^{(\iota)}$ are trivial for $j = 1, 2$ if $\iota \in I_c$. For $\iota \in I_{nc}$, the weights of ν on $W_1^{(\iota)}$ are of the form $\frac{k-2}{4} - m_1$ and those on $W_2^{(\iota)}$ are of the form $\frac{k}{4} - m_2$ with $m_1, m_2 \in \mathbf{Z}$. If k is even m_1 runs from 0 to $\frac{k-2}{2}$ and m_2 from 0 to $\frac{k}{2}$. For k odd, both m_1 and m_2 run from 0 to $\frac{k-1}{2}$.

Proof. The decomposition of W^s induces the decomposition (5.3.$*$). The quasi-cocharacter ν projects trivially to the factors of $\widetilde{G}^{\mathrm{der}}_{/\mathbf{C}}$ corresponding to the $\iota \in I_c$ and non-trivially to the other factors. The highest weights of ν on $W_1^{(\iota)}$ and $W_2^{(\iota)}$ can easily be deduced from the information collected in [Del79, Table 1.3.9], it is the rational number corresponding to ϖ_{k-1} resp. ϖ_k in that table. The lowest weight of ν on $W_1^{(\iota)}$ is the opposite of the number corresponding to ϖ_{k-1} if k is even and the opposite of the number corresponding to ϖ_k if k is odd. For $W_2^{(\iota)}$, the converse is the case. $\square$

We continue the construction of $\widetilde{G}$ and $\tilde{\mu}$. The case where k is even and the case where k is odd will be treated separately.

5.4 The case where k is even. In this case, K is totally real and either $K = K_0$ or $[K : K_0] = 2$. Let L' be a totally imaginary quadratic extension of K_0, put $L = KL'$ and define the algebraic torus T_L over $\mathbf{Q}$ by $T_L = \ker(N_{L/K}) \subset L^\times$,

where $N_{L/K}\colon L^\times \to K^\times$ is the field norm. The group T_L naturally acts on L and this gives rise to a $\mathbf{Q}$-linear representation V_L of T_L. We have a natural structure of K-vector space on V_L. For each embedding $\iota\colon K \hookrightarrow \mathbf{C}$ we choose a complex embedding of L extending ι and this gives an identification

$$T_{L/\mathbf{C}} = \bigoplus_{\iota \in \widetilde{I}} \mathbf{G}_{m/\mathbf{C}},$$

where $\widetilde{I}$ is the set of complex embeddings of K.

From now on, we will further distinguish the cases where $[K : K_0] = 1$ and where $[K : K_0] = 2$.

The subcase where $K = K_0$**.** Consider the decomposition (5.3.$*$) of $W \otimes_{\mathbf{Q}} \mathbf{C}$. Since $W^s = W_1^s \oplus W_2^s$ is a decomposition of W^s as a direct sum of K_0-vector spaces, we obtain a $\mathbf{Q}$-linear decomposition $W = W_1 \oplus W_2$. For $\iota \in I_{nc}$, the weights of ν on $W_1^{(\iota)}$ are in $\frac{1}{2} + \mathbf{Z}$ if $k \equiv 0 \pmod 4$ and in $\mathbf{Z}$ if $k \equiv 2 \pmod 4$. For the weights on $W_2^{(\iota)}$, the converse is the case.

Let $T = T_L \times T_L$ and define W_3 (resp. W_4) be the representation of T given by the action of the first (resp. the second) factor T_L on V_L. We define a quasi-cocharacter

$$\nu_L\colon \mathbf{G}_{m/\mathbf{C}} \to T_{/\mathbf{C}} = \bigoplus_{\iota \in I} \mathbf{G}^2_{m/\mathbf{C}},$$

by

$$\nu_L(z)_\iota = \begin{cases} (1,1) & \text{if } \iota \in I_c \\ (\sqrt{z}, 1) & \text{if } \iota \in I_{nc} \text{ and } k \equiv 0 \pmod 4 \\ (1, \sqrt{z}) & \text{if } \iota \in I_{nc} \text{ and } k \equiv 2 \pmod 4), \end{cases}$$

where $\nu_L(z)_\iota$ is the component of $\nu_L(z)$ in the factor of $T_{/\mathbf{C}}$ corresponding to $\iota \in I$. Finally, let $\widetilde{V}$ be the representation of $\widetilde{G}^{\mathrm{der}} \times T$ defined by

$$\widetilde{V} = W_1 \otimes_{K_0} W_3 \oplus W_2 \otimes_{K_0} W_4$$

and let $\widetilde{G}$ be the image of $\widetilde{G}^{\mathrm{der}} \times T$ in $\mathrm{GL}(\widetilde{V})$. This will not cause any confusion, since $\widetilde{G}^{\mathrm{der}}$ acts faithfully on $\widetilde{V}$, so the derived group of $\widetilde{G}$ is $\widetilde{G}^{\mathrm{der}}$. The weights of (ν, ν_L) on $\widetilde{V}$ are in $\mathbf{Z}$, so the projection of the quasi-cocharacter (ν, ν_L) from $(\widetilde{G}^{\mathrm{der}} \times T)_{/\mathbf{C}}$ to $\widetilde{G}_{/\mathbf{C}}$ is a true cocharacter $\tilde{\mu}$ of $\widetilde{G}_{/\mathbf{C}}$. By construction,

$$\tilde{\pi}^{\mathrm{ad}} \circ \tilde{\mu} = \tilde{\pi}^{\mathrm{ad}} \circ \nu = \pi^{\mathrm{ad}} \circ \mu,$$

so we have constructed the couple $(\widetilde{G}, \tilde{\mu})$ in this particular subcase.

The subcase where $[K : K_0] = 2$. The structure of K-vector space on W gives rise to a decomposition
$$W \otimes_{\mathbf{Q}} \mathbf{C} \cong \bigoplus_{\tilde{\iota}\colon K \hookrightarrow \mathbf{C}} W^{(\tilde{\iota})}. \tag{5.4.$*$}$$
Recall that $\widetilde{I}$ is the set of complex embeddings of K. Put $\widetilde{I}_c = \{\tilde{\iota} \in \widetilde{I} \mid \tilde{\iota}_{|K_0} \in I_c\}$ and $\widetilde{I}_{nc} = \{\tilde{\iota} \in \widetilde{I} \mid \tilde{\iota}_{|K_0} \in I_{nc}\}$. For $\tilde{\iota} \in \widetilde{I}_c$, the weights of ν on $W^{(\tilde{\iota})}$ are all 0, for $\tilde{\iota} \in \widetilde{I}_{nc}$, these weights are either in $\mathbf{Z}$ or in $\frac{1}{2} + \mathbf{Z}$. Let $\widetilde{I}_{nc,1}$ be the set of $\tilde{\iota} \in \widetilde{I}_{nc}$ where the former possibility occurs and $\widetilde{I}_{nc,2}$ its complement in $\widetilde{I}_{nc}$. Note that for any $\iota \in I_{nc}$ one of the embeddings $K \hookrightarrow \mathbf{C}$ restricting to ι lies in $\widetilde{I}_{nc,1}$ and the other one lies in $\widetilde{I}_{nc,2}$.

This time we define a quasi-cocharacter
$$\nu_L\colon \mathbf{G}_{m/\mathbf{C}} \to T_{L/\mathbf{C}} = \bigoplus_{\tilde{\iota} \in \widetilde{I}} \mathbf{G}_{m/\mathbf{C}},$$
by
$$\nu_L(z)_{\tilde{\iota}} = \begin{cases} 1 & \text{if } \tilde{\iota} \in \widetilde{I}_c \cup \widetilde{I}_{nc,1} \\ \sqrt{z} & \text{if } \tilde{\iota} \in \widetilde{I}_{nc,2}. \end{cases}$$
As above, $\nu_L(z)_{\tilde{\iota}}$ is the component of $\nu_L(z)$ in the factor of $T_{L/\mathbf{C}}$ corresponding to $\tilde{\iota} \in \widetilde{I}$. Finally, let $\widetilde{V} = W \otimes_K V_L$ as representation of $\widetilde{G}^{\mathrm{der}} \times T$ and let $\widetilde{G}$ be the image of $\widetilde{G}^{\mathrm{der}} \times T$ in $\mathrm{GL}(\widetilde{V})$. Once again, there is no risk of confusion, because the derived group of $\widetilde{G}$ is $\widetilde{G}^{\mathrm{der}}$. Since the weights of (ν, ν_L) on $\widetilde{V}$ are in $\mathbf{Z}$, the projection of (ν, ν_L) from $(\widetilde{G}^{\mathrm{der}} \times T)_{/\mathbf{C}}$ to $\widetilde{G}_{/\mathbf{C}}$ is a true cocharacter $\tilde{\mu}$ of $\widetilde{G}_{/\mathbf{C}}$. By construction, $\tilde{\pi}^{\mathrm{ad}} \circ \tilde{\mu} = \tilde{\pi}^{\mathrm{ad}} \circ \nu = \pi^{\mathrm{ad}} \circ \mu$, so this achieves the construction of the couple $(\widetilde{G}, \tilde{\mu})$ in the case where k is even.

5.5 The case where k is odd. In this case, K is a totally imaginary quadratic extension of K_0. In order to unify this case with the previous one as much as possible, we put $L = K$. Let T_L be the $\mathbf{Q}$-algebraic torus $\ker(N_{L/K_0})$, where $N_{L/K_0}\colon L^\times \to K_0^\times$ is the field norm. As in the beginning of 5.4, the action of T_L on L by multiplication on the left gives rise to a representation V_L of T_L.

We again consider the decomposition (5.4.$*$) and the subsets $\widetilde{I}_c$ and $\widetilde{I}_{nc}$ of the set $\widetilde{I}$ of complex embeddings of $K = L$. By lemma 5.3, the weights of ν on $W^{(\tilde{\iota})}$ are either in $\frac{1}{4} + \mathbf{Z}$ or in $-\frac{1}{4} + \mathbf{Z}$ for $\tilde{\iota} \in \widetilde{I}_{nc}$ and trivial for $\tilde{\iota} \in \widetilde{I}_c$. Let $\widetilde{I}_{nc,1}$ the set of $\tilde{\iota} \in \widetilde{I}_{nc}$ for which the weights of ν on $W^{(\tilde{\iota})}$ are in $\frac{1}{4} + \mathbf{Z}$ and let $\widetilde{I}_{nc,2}$ be the complement of $\widetilde{I}_{nc,1}$ in $\widetilde{I}_{nc}$. For each $\iota \in I_{nc}$, one of the embeddings

$K \hookrightarrow \mathbf{C}$ restricting to ι lies in $\widetilde{I}_{nc,1}$ and the other one lies in $\widetilde{I}_{nc,2}$. Thus, the map $r\colon \widetilde{I} \to I$ given by $\tilde{\iota} \mapsto \tilde{\iota}_{|K_0}$ induces a bijection of $\widetilde{I}_{nc,1}$ with I_{nc}. For I_c, we arbitrarily fix a subset $\widetilde{I}_{c,1} \subset \widetilde{I}_c$ such that r induces a bijection of $\widetilde{I}_{c,1}$ with I_c. Putting $\widetilde{I}_1 = \widetilde{I}_{nc,1} \cup \widetilde{I}_{c,1}$, we get an identification

$$T_{L/\mathbf{C}} = \bigoplus_{\tilde{\iota} \in \widetilde{I}_1} \mathbf{G}_{m/\mathbf{C}}.$$

Under this identification the factor $\mathbf{G}_{m/\mathbf{C}}$ corresponding to $\tilde{\iota}$ acts on the factor of $V_{L/\mathbf{C}}$ corresponding to $\tilde{\iota}$ by multiplication in case $\tilde{\iota} \in \widetilde{I}_1$ and by multiplication with the inverse if $\tilde{\iota} \notin \widetilde{I}_1$.

The quasi-cocharacter $\nu_L\colon \mathbf{G}_{m/\mathbf{C}} \to T_{L/\mathbf{C}}$ is defined by

$$\nu_L(z)_{\tilde{\iota}} = \begin{cases} 1 & \text{if } \tilde{\iota} \in \widetilde{I}_{c,1} \\ z^{-1/4} & \text{if } \tilde{\iota} \in \widetilde{I}_{nc,1}. \end{cases}$$

We put $\widetilde{V} = W \otimes_L V_L$ as representation of $\widetilde{G}^{\mathrm{der}} \times T$ and again let $\widetilde{G}$ be the image of $\widetilde{G}^{\mathrm{der}} \times T$ in $\mathrm{GL}(\widetilde{V})$. As before, the derived group of $\widetilde{G}$ is $\widetilde{G}^{\mathrm{der}}$ and the projection of (ν, ν_L) from $(\widetilde{G}^{\mathrm{der}} \times T)_{/\mathbf{C}}$ to $\widetilde{G}_{/\mathbf{C}}$ is a true cocharacter $\tilde{\mu}$ of $\widetilde{G}_{/\mathbf{C}}$. We obviously have equalities $\tilde{\pi}^{\mathrm{ad}} \circ \tilde{\mu} = \tilde{\pi}^{\mathrm{ad}} \circ \nu = \pi^{\mathrm{ad}} \circ \mu$, so this settles the construction of the couple $(\widetilde{G}, \tilde{\mu})$ in this case.

The preceding discussion establishes the following theorem.

5.6 Theorem. *Let $A/\mathbf{C}$ be a simple abelian variety, (G_A, h_A) its Mumford–Tate datum and μ_A the Hodge cocharacter associated to A. Assume that A is essentially Mumford–Tate unliftable and that G_A is of type $D_k^{\mathbf{H}}$ with $k \geq 4$. Then there exist an algebraic group $\widetilde{G}$ with $\widetilde{G}^{\mathrm{der}}$ simply connected, an identification $\widetilde{G}^{\mathrm{ad}} = G_A^{\mathrm{ad}}$ and a cocharacter $\tilde{\mu}\colon \mathbf{G}_{m/\mathbf{C}} \to \widetilde{G}_{/\mathbf{C}}$ such that $\tilde{\pi}^{\mathrm{ad}} \circ \tilde{\mu} = \pi^{\mathrm{ad}} \circ \mu_A$.*

5.7 Remark. In all cases above, the group $T_{L/\mathbf{R}}$ occurring in the construction of $\widetilde{G}$ is compact. It thus follows that $\widetilde{G}^{\mathrm{ab}}_{/\mathbf{R}}$ is compact. It is left to the reader to construct an isomorphism $\widetilde{G}^{\mathrm{ab}} \cong T_L$ and to compute the composite $T_L \subset \widetilde{G} \to \widetilde{G}^{\mathrm{ab}} \cong T_L$.

5.8 Suppose that $F \subset \mathbf{C}$ is a number field and that A is an abelian variety over F with connected Mumford–Tate group such that the conditions of the theorem

are verified for $A_{/\mathbf{C}}$ and its Mumford–Tate datum (G_A, h_A). Let $\widetilde{G}$ and $\tilde{\mu}$ be as in the conclusion of the theorem.

Let $\widetilde{G}^{\mathrm{ab}} = \widetilde{G}/\widetilde{G}^{\mathrm{der}}$ and define μ^{ab} to be the composite of the cocharacter $\tilde{\mu}\colon \mathbf{G}_{m/\mathbf{C}} \to \widetilde{G}_{/\mathbf{C}}$ with the natural projection $\widetilde{G}_{/\mathbf{C}} \to \widetilde{G}^{\mathrm{ab}}_{/\mathbf{C}}$. Define

$$\begin{aligned} h^{\mathrm{ab}}\colon S &\to \widetilde{G}^{\mathrm{ab}}_{/\mathbf{R}} \\ z &\mapsto \mu^{\mathrm{ab}}(z)\overline{\mu^{\mathrm{ab}}(\bar{z})}. \end{aligned}$$

The remark 5.7 implies that $\widetilde{G}^{\mathrm{ab}}_{/\mathbf{R}}$ is compact. As the weight $w_h\colon \mathbf{G}_{m/\mathbf{R}} \to \widetilde{G}^{\mathrm{ab}}_{/\mathbf{R}}$ is trivial, [Del79, 1.1.18(b)] implies that h^{ab} defines a polarizable Hodge structure on any $\mathbf{Q}$-linear representation V^{ab} of $\widetilde{G}^{\mathrm{ab}}$.

Fix a faithful rational representation V^{ab} of $\widetilde{G}^{\mathrm{ab}}$. It follows from proposition A.1 of [Del82b] and the remark preceding it that the Hodge structure on V^{ab} is the Betti realization of an absolute Hodge motive $M^{\mathrm{ab}}_{/\overline{\mathbf{Q}}}$ belonging to $(\mathbf{CM})_{\overline{\mathbf{Q}}}$. Note that using the isomorphism $\widetilde{G}^{\mathrm{ab}} \cong T_L$ from remark 5.7, this motive can also be constructed explicitly. Replacing F by a finite extension and fixing an embedding $F \subset \overline{\mathbf{Q}}$ we can assume that $M^{\mathrm{ab}}_{/\overline{\mathbf{Q}}}$ descends to a motive M^{ab} over F. At the cost of a further finite extension of F, we can also assume that the Mumford–Tate group of M^{ab} is connected and hence contained in $\widetilde{G}^{\mathrm{ab}}$. This implies that for every p, the p-adic realization of M^{ab} factors through a map $\rho^{\mathrm{ab}}_p\colon \mathscr{G}_F \to \widetilde{G}^{\mathrm{ab}}(\mathbf{Q}_p)$. Note that the ρ^{ab}_p form a compatible system of representations (see 6.7) and that the ρ^{ab}_p do not depend on the choice of the representation V^{ab} of $\widetilde{G}^{\mathrm{ab}}$.

For each prime number p, let Σ_p be the set of p-adic places of F. There is a finite set Σ of places v of F such that A and M^{ab} have good reduction outside Σ. For every prime number p, the representations $\rho_{A,p}$ and $\tilde{\rho}^{\mathrm{ab}}_p$ are unramified at all $v \notin \Sigma \cup \Sigma_p$. For any finite extension F' of F, let Σ'_p be the set of p-adic places of F' and let Σ' be the set of places lying over the $v \in \Sigma$.

Let $G' = G^{\mathrm{ad}}_A \times \widetilde{G}^{\mathrm{ab}}$ and let $\pi'\colon \widetilde{G} \to G'$ be defined by the projections of $\widetilde{G}$ onto $\widetilde{G}^{\mathrm{ab}}$ and $G^{\mathrm{ad}}_A = \widetilde{G}^{\mathrm{ad}}$. For each p, define $\rho^{\mathrm{ad}}_p = \pi^{\mathrm{ad}} \circ \rho_{A,p}\colon \mathscr{G}_F \to G^{\mathrm{ad}}_A(\mathbf{Q}_p)$ and put

$$\rho'_p = (\rho^{\mathrm{ad}}_p, \rho^{\mathrm{ab}}_p)\colon \mathscr{G}_F \to G'(\mathbf{Q}_p).$$

5.9 Corollary. *There exist a finite extension F' of F and a system of geometric Galois representations $\tilde{\rho}_p\colon \mathscr{G}_{F'} \to \widetilde{G}(\mathbf{Q}_p)$ lifting the restrictions $\rho'_p\colon \mathscr{G}_{F'} \to G'(\mathbf{Q}_p)$. The system $(\tilde{\rho}_p)$ can be chosen in such a way that each $\tilde{\rho}_p$ is unramified at v for each $v \notin \Sigma' \cup \Sigma'_p$.*

For each prime number p, the representation $\tilde{\rho}_p$ is a weak geometric lift of $\rho_{A,p}$ and it is essentially geometrically unliftable.

Proof. Since G_A is the Mumford–Tate group of A, the adjoint representation of G_A gives rise to a motive M^{ad} belonging to the subcategory of $\mathbf{Mot}_{\mathrm{AH}}(F)$ which is $\otimes$-generated by $h^1(A)$ and the Tate motive. The Mumford–Tate group of M^{ad} is the adjoint group $G_A^{\mathrm{ad}} = \widetilde{G}^{\mathrm{ad}}$.

Consider the object $M' = M^{\mathrm{ad}} \times M^{\mathrm{ab}}$ of $\mathbf{Mot}_{\mathrm{AH}}(F)$. Its Mumford–Tate group is contained in G' and the Hodge cocharacter $\mu' \colon \mathbf{G}_{m/\mathbf{C}} \to G'_{/\mathbf{C}}$ associated to its Betti realization is the product $(\mu^{\mathrm{ad}}, \mu^{\mathrm{ab}})$. It follows that $\mu' = \pi'_{/\mathbf{C}} \circ \tilde{\mu}$. On the other hand, for each p, the Galois representation on the p-adic realization of the motive M' is the above map ρ'_p. The motive M' is an abelian motive, so it follows from [Bla94, Theorem 0.3] that the p-adic comparison maps linking the p-adic and the DeRham realizations of M' are compatible with absolute Hodge classes. The main theorem 2.1.7 of [Win95] implies that there is a finite extension $F' \supset F$ such that every $(\rho'_p)_{|\mathscr{G}_{F'}}$ lifts to a representation $\tilde{\rho}_p \colon \mathscr{G}_{F'} \to \widetilde{G}(\mathbf{Q}_p)$ satisfying the conditions of the corollary.

It is clear from the above that, for each p, the map $\tilde{\rho}_p$ provides a weak geometric lift of $\rho_{A,p}$. As $\widetilde{G}^{\mathrm{der}}$ is simply connected, it is also clear that the $\tilde{\rho}_p$ are essentially geometrically unliftable. ☐

5.10 Definition. The representations $\tilde{\rho}_p$ constructed as in the corollary from Mumford–Tate decomposed abelian varieties will be called *representations of lifted abelian $D_k^{\mathbf{H}}$-type*.

5.11 Corollary. *Let $F \subset \mathbf{C}$ be a number field and A/F an abelian variety with connected Mumford–Tate group G_A and associated system of Galois representations $(\rho_{A,p})$. Then there exist an linear algebraic group $\widetilde{G}$ over $\mathbf{Q}$ such that $\widetilde{G}^{\mathrm{der}}$ is the universal cover of G_A^{der}, a finite extension F' of F and a system of weak geometric lifts*

$$\tilde{\rho}_p \colon \mathscr{G}_{F'} \to \widetilde{G}(\mathbf{Q}_p)$$

of the restrictions to $\mathscr{G}_{F'}$ of the $\rho_{A,p}$.

Proof. By theorem 2.12 and proposition 3.1, there exist a number field F' and an essentially M-T unliftable and M-T decomposed weak Mumford–Tate lift B/F' of A with connected Mumford–Tate group G_B. It follows from proposition 4.5

that the system of Galois representations $(\rho_{B,p})$ associated to B is a system of weak geometric lifts of the system $(\rho_{A,p})$.

Let $B_{/\mathbf{C}} \sim \prod B_{i/\mathbf{C}}$ be the isogeny decomposition as in definition 2.10. After enlarging F', we can assume that this decomposition exists over F'. In that case, each B_i is Mumford–Tate decomposed with $G_{B_i}^{\mathrm{ad}}$ simple and G_B^{ad} is the product of the $G_{B_i}^{\mathrm{ad}}$. For each i, let $\rho_{B_i,p}\colon \mathscr{G}_{F'} \to G_{B_i}(\mathbf{Q}_p)$ be the p-adic Galois representation associated to B_i. For every i such that G_{B_i} is not of type $D_k^{\mathbf{H}}$, put $\widetilde{G}_i = G_{B_i}$ and $\tilde{\rho}_{i,p} = \rho_{B_i,p}$. For each i such that G_{B_i} is of type $D_k^{\mathbf{H}}$ let $\tilde{\rho}_{i,p}\colon \mathscr{G}_{F'} \to \widetilde{G}_i(\mathbf{Q}_p)$ be the representation of lifted abelian $D_k^{\mathbf{H}}$-type resulting from corollary 5.9, replacing F' by a finite extension again if necessary. Let $\widetilde{G} = \prod \widetilde{G}_i$ and let $\tilde{\rho}_p\colon \mathscr{G}_{F'} \to \widetilde{G}(\mathbf{Q}_p)$ be the product of the maps $\tilde{\rho}_{i,p}$.

For each prime number p, the representation $\tilde{\rho}_p$ is a weak geometric lift of $\rho_{B,p}$ and therefore also of $\rho_{A,p}$. As $\widetilde{G}^{\mathrm{der}}$ is simply connected, the corollary follows. $\square$

5.12 Corollary. *Let F be a number field, M an abelian motive and $\rho'\colon \mathscr{G}_F \to G'(\mathbf{Q}_p)$ a weak geometric lift of the associated p-adic Galois representation. Then, after replacing F by a finite extension and for any linear representation V'_p of $G'_{/\mathbf{Q}_p}$, the representation of $\mathscr{G}_F$ on V'_p deduced from ρ'_p lies in the tannakian subcategory of $\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$ generated by the p-adic representations associated to abelian motives and the representations of lifted abelian $D_k^{\mathbf{H}}$-type.*

Proof. It is sufficient to prove the corollary for one fixed faithful self-dual representation V'_p of G'.

Let A be the essentially Mumford–Tate unliftable and Mumford–Tate decomposed abelian variety supplied by applying corollary 3.6 to M, so M is an object of $\langle h^1(A), \mathbf{Q}(1)\rangle$. This inclusion corresponds to a map $G_A \to G_M$. Suppressing isogeny factors of A, we can assume that the induced map $G_A^{\mathrm{der}} \to G_M^{\mathrm{der}}$ is an isogeny. After replacing F by a finite extension, define the group $\widetilde{G}$ and the representation $\tilde{\rho}_p\colon \mathscr{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ as in the proof of 5.11.

As $\widetilde{G}^{\mathrm{der}}$ is the simply connected cover of G_A^{der}, the isogeny

$$\widetilde{G}^{\mathrm{der}} \to G_A^{\mathrm{der}} \to G_M^{\mathrm{der}}$$

lifts to an isogeny $\widetilde{G}^{\mathrm{der}} \to G'^{\mathrm{der}}$. This results in an identification of the adjoint groups $G'^{\mathrm{ad}} = G_M^{\mathrm{ad}} = G_A^{\mathrm{ad}} = \widetilde{G}^{\mathrm{ad}}$ and, by construction, the projections to $G_M^{\mathrm{ad}}(\mathbf{Q}_p)$ of the representations ρ'_p, $\rho_{M,p}$, $\rho_{A,p}$ and $\tilde{\rho}_p$ all coincide. The corollary

now follows from lemma 4.7 with $G_1 = \widetilde{G}$, V_1 any faithful self-dual representation of G_1, $G_2 = G'$ and $V_2 = V'_p$. □

6 Galois representations of lifted abelian $D_k^{\mathbf{H}}$-type

6.1 We revert to the notations of 5.8, in particular A/F is essentially Mumford–Tate unliftable, G_A is of type $D_k^{\mathbf{H}}$ and $\widetilde{G}$ is the group constructed in theorem 5.6. We will also assume A to be Mumford–Tate decomposed. For the first part of this section, we fix a prime number p and assume that there exists a weak geometric lift $\tilde{\rho}_p \colon \mathscr{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ of $\rho_{A,p}$ with $\widetilde{G}$ as in theorem 5.6. It follows from corollary 5.9 that such a lifting exists after replacing F by a finite extension and that, for F big enough, $\tilde{\rho}_p$ is part of a system of p-adic representations for variable p, but we do not assume for the moment that such a system exists.

6.2 Theorem. *Let $\tilde{\rho}_p \colon \mathscr{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ be a representation of lifted abelian $D_k^{\mathbf{H}}$-type, constructed as weak geometric lift of the representation $\rho_{A,p}$ as described in 6.1, with $k \geq 5$. Assume moreover that $\rho_{A,p}(\mathscr{G}_F) \subset G_A(\mathbf{Q}_p)$ is Zariski dense.*

Let $\widetilde{V}_p$ be a faithful $\mathbf{Q}_p$-linear representation of $\widetilde{G}_{/\mathbf{Q}_p}$. Then there is no finite extension F' of F such that the representation of $\mathscr{G}_{F'}$ on $\widetilde{V}_p$ induced by $\tilde{\rho}_p$ belongs to the category $(\mathbf{AV})$-$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{F'})$.

Proof. Replace F by a finite extension and assume that the representation of $\mathscr{G}_F$ on $\widetilde{V}_p$ belongs to $(\mathbf{AV})$-$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_F)$. After further enlarging F and applying proposition 4.5 and remark 4.10, it follows that there is an essentially Mumford–Tate unliftable and Mumford–Tate decomposed abelian variety B/F such that $\widetilde{V}_p$ belongs to $\langle \mathrm{H}^1_{\text{ét}}(B_{\bar{F}}, \mathbf{Q}_p), (\mathbf{CM})\text{-}\mathbf{Rep} \rangle$.

Let $B \sim \prod_{j=1}^{m-1} B_j$ be the decomposition from the definition 2.10, so that $G_B^{\mathrm{der}} = \prod G_{B_j}^{\mathrm{der}}$, each $G_{B_j}^{\mathrm{ad}}$ is simple and $G_B^{\mathrm{ad}} = \prod G_{B_j}^{\mathrm{ad}}$. For $j = 1, \cdots, m-1$, let $\rho_j = \rho_{B_j,p}$, put $V_j = \mathrm{H}^1_{\text{ét}}(B_{j/\bar{F}}, \mathbf{Q}_p)$ and let H_j be the Zariski closure of $\rho_j(\mathscr{G}_F)$ in G_{B_j}. Let V_m be an object of $(\mathbf{CM})$-$\mathbf{Rep}$ such that $\widetilde{V}_p$ belongs to $\langle V_1, \ldots, V_m \rangle$, let H_m be the Zariski closure of the image of the Galois representation on V_m and let $\rho_m \colon \mathscr{G}_F \to H_m(\mathbf{Q}_p)$ be the corresponding morphism. Finally, let $H \subset \prod_{j=1}^{m} H_j$ be the Zariski closure of the image of

$$(\rho_1, \ldots, \rho_m) \colon \mathscr{G}_F \to \prod_{j=1}^{m} H_j(\mathbf{Q}_p)$$

and $\sigma\colon \mathscr{G}_F \to H(\mathbf{Q}_p)$ the induced representation. At the cost of a further finite extension of F, we can assume H to be connected. The fact that $\widetilde{V}_p$ belongs to the tannakian category generated by the V_j implies that there is a surjection $\pi_p\colon H \to \widetilde{G}_{/\mathbf{Q}_p}$ such that $\tilde{\rho}_p = \pi_p \circ \sigma$.

For any p-adic place $\bar{v}$ of $\bar{F}$, the cocharacter $\mu_{\tilde{\rho}_p,\bar{v}}\colon \mathbf{G}_{m/\mathbf{C}_p} \to \widetilde{G}_{/\mathbf{C}_p}$ associated to the Hodge–Tate decomposition corresponding to $(\tilde{\rho}_p)_{|\mathscr{I}_{\bar{v}}}$ lifts to the cocharacter $\mu_{\sigma,\bar{v}}\colon \mathbf{G}_{m/\mathbf{C}_p} \to H_{/\mathbf{C}_p}$ associated to the Hodge–Tate decomposition corresponding to $\sigma_{|\mathscr{I}_{\bar{v}}}$. There is at least one simple factor $G'_{/\mathbf{C}_p}$ of $\widetilde{G}^{\mathrm{ad}}_{/\mathbf{C}_p}$ to which $\mu_{\tilde{\rho}_p,\bar{v}}$ projects non-trivially. It follows from [Win88, Proposition 7] and the fact that [Bla94, Theorem 0.3] implies [Win88, Conjecture 1] that this projection is dual to the root α_k of this factor.

Let $H'_{/\mathbf{C}_p}$ be the simple isogeny factor of $H^{\mathrm{der}}_{/\mathbf{C}_p}$ which surjects onto $G'_{/\mathbf{C}_p}$, such a factor exists by hypothesis. As $\widetilde{G}^{\mathrm{der}}$ is simply connected, $H'_{/\mathbf{C}_p}$ is simply connected as well. The Hodge–Tate cocharacter $\mu_{\sigma,\bar{v}}$ lifts to a quasi-cocharacter $\mu'_{\bar{v}}$ of $H'_{/\mathbf{C}_p}$. This quasi-cocharacter is still dual to the vertex α_k of the Dynkin diagram. Over $\mathbf{C}_p$, any faithful representation W' of $H'_{/\mathbf{C}_p}$ whose highest weights are fundamental weights contains a direct factor with highest weight ϖ_{k-1} or ϖ_k and it follows from lemma 5.3 applied to $H'_{/\mathbf{C}_p}$, with $\mu'_{\bar{v}}$ playing the role of ν, that $\mu'_{\bar{v}}$ has at least three weights on this factor. We will show that this leads to a contradiction.

As $W = \bigoplus_{j=1}^{m-1} V_j$ is the Galois representation on $\mathrm{H}^1_{\text{ét}}(B_{\bar{F}}, \mathbf{Q}_p)$, the Hodge–Tate cocharacter $\mu_{\sigma,\bar{v}}$ acts on $W \otimes \mathbf{C}_p$ with two weights. By construction, W is a faithful representation of H^{der} so $W \otimes_{\mathbf{Q}_p} \mathbf{C}_p$ is a faithful representation of $H'_{/\mathbf{C}_p}$. Let W' be the direct sum of the direct factors of the representation of $H'_{/\mathbf{C}_p}$ on $W \otimes \mathbf{C}_p$ on which $H'_{/\mathbf{C}_p}$ acts non-trivially. The quasi-cocharacter $\mu'_{\bar{v}}$ then acts with exactly two weights on W'. It follows from [Del79, 1.3.7] that for every irreducible direct factor W'' of W', the highest weight is a fundamental weight of $H'_{/\mathbf{C}_p}$. We have shown above that this implies that $\mu'_{\bar{v}}$ has at least three weights on W'', so we arrive at the contradiction we were looking for. □

6.3 Remark. It follows from theorems 2.9 and 2.12 and their proof, that for every $k \geq 4$ there exists an essentially M-T unliftable and M-T decomposed abelian variety $A/\mathbf{C}$ such that G_A is of type $D_k^{\mathbf{H}}$. It follows from [Noo95, Theorem 1.7] that there also exists such an abelian variety which can be defined over a number field F and for which $\rho_{A,p}(\mathscr{G}_F) \subset G_A(\mathbf{Q}_p)$ is Zariski dense. Thus, for

every $k \geq 5$, there is an abelian variety for which the hypothesis of the theorem is fulfilled.

6.4 The case $k = 4$. We consider the case of Galois representations of lifted abelian $D_4^{\mathbf{H}}$-type, so we keep all the notations of 6.1 but fix $k = 4$. In this case, the method of the proof of the above theorem gives a more limited statement. The reason for this lies in the facts that the representation of $\mathscr{G}_F$ on $\widetilde{V}_p$ may be reducible and that the group $\widetilde{G}_{/\mathbf{C}}$ does have faithful representations in which the Hodge cocharacter acts with only two weights.

It is left to the reader to verify the following statement, whose proof is completely analogous to the proof of theorem 6.2.

6.5 Proposition. *Let notations and hypotheses be as in theorem 6.2, but with $k = 4$.*

Let $\widetilde{V}$ be a faithful $\mathbf{Q}$-linear representation of $\widetilde{G}$. Then there is no finite extension F' of F such that the representation of $\mathscr{G}_{F'}$ on $\widetilde{V} \otimes \mathbf{Q}_p$ induced by $\tilde{\rho}_p$ belongs to the category $(\mathbf{AV})$-$\mathbf{Rep}_{\mathbf{Q}_p}(\mathscr{G}_{F'})$.

6.6 Remark. In the situation of the proposition, assume that p is a prime number such that the image of $\mathscr{G}_{\mathbf{Q}_p}$ in the automorphism group of the Dynkin diagram is equal to the image of $\mathscr{G}_{\mathbf{Q}}$ in this automorphism group. Then the statement of the proposition (and of theorem 6.2) holds for every $\mathbf{Q}_p$-linear representation $\widetilde{V}_p$ of $\widetilde{G}_{/\mathbf{Q}_p}$. If $\mathscr{G}_{\mathbf{Q}}$ acts on the Dynkin diagram of $G_{\overline{\mathbf{Q}}}$ through a cyclic group, then the images of $\mathscr{G}_{\mathbf{Q}}$ and $\mathscr{G}_{\mathbf{Q}_p}$ in the automorphism group of the Dynkin diagram coincide for infinitely many p.

On the other hand, there also exist examples of Galois representations of lifted abelian $D_4^{\mathbf{H}}$-type where the proof of the theorem fails for every p. One can find an example of this situation by taking an abelian variety for which $\mathscr{G}_{\mathbf{Q}}$ acts on the Dynkin diagram of the Mumford–Tate group through $(\mathbf{Z}/2\mathbf{Z})^2$. For every prime number p, the image of $\mathscr{G}_{\mathbf{Q}_p}$ in the automorphism group of the Dynkin diagram is then of order at most 2.

6.7 Frobenius elements. Let v be a valuation of F. By $\mathrm{Fr}_v \in \mathscr{G}_F$, we will denote a geometric Frobenius element at v, i. e. an element of the decomposition group of a place $\bar{v}$ of $\bar{F}$ lying over v such that, on the residue field $\bar{k}_v$ of $\bar{F}$ at $\bar{v}$, Fr_v induces the *inverse* of the map $x \mapsto x^{q_v}$, where q_v is the order of the residue

field k_v of F a v. Note that Fr_v is defined only up to conjugation and up to multiplication by an element of $\mathscr{I}_{F,\bar{v}}$. This implies that the image of Fr_v in a representation which is unramified at v is defined up to conjugation and that the eigenvalues and the characteristic polynomial of the image of Fr_v in such a representation are well defined.

Let V_p be a $\mathbf{Q}_p$-linear representation of $\mathscr{G}_F$ on an étale cohomology group of a proper and smooth F-variety and let v be a place where this variety has good reduction. It follows from the Weil conjectures, proved by Deligne, that the eigenvalues of a Frobenius element Fr_v at v are algebraic integers and that all complex absolute values of all these eigenvalues coincide. If the Fontaine–Mazur conjecture is true, then every geometric representation of $\mathscr{G}_F$ should have this property. We will verify this for the representations of lifted abelian $D_k^{\mathbf{H}}$-type. It is also shown that Fr_v acts semi-simply in any representation of lifted abelian $D_k^{\mathbf{H}}$-type, a property which is conjectured for all representations coming from the cohomology of algebraic varieties.

The Weil conjectures also imply that, for varying p, the p-adic étale cohomology groups of a proper and smooth variety form a *compatible system* (V_p) of Galois representations. This means that there is a finite set Σ of valuations of F such that V_p is unramified at v for all $v \notin \Sigma$ with $v(p) = 0$ and that for all $v \notin \Sigma$ there is a polynomial $P^v \in \mathbf{Q}[X]$ which is equal to the characteristic polynomial of Fr_v acting on V_p for all p with $v(p) = 0$. We will prove a result in this direction for the representations of lifted abelian $D_k^{\mathbf{H}}$-type.

It should be pointed out that this property does not follow from the conjecture of Fontaine and Mazur. However, combined with the Mumford–Tate conjecture, the Fontaine–Mazur conjecture implies that for any geometric p-adic representation W_p of $\mathscr{G}_F$, there should exist a number field E and a compatible system (indexed by the primes $\mathfrak{p}$ of E) of $E_{\mathfrak{p}}$-linear representations of $\mathscr{G}_F$ such that W_p occurs in this system. The characteristic polynomials of Fr_v acting on the V_p would then lie in $E[X]$ and be independent of p, for all p with $v(p) = 0$.

6.8 Proposition. *Let $F \subset \mathbf{C}$ be a number field and let $\tilde{\rho}_p \colon \mathscr{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ be a representation of lifted abelian $D_k^{\mathbf{H}}$-type, with $k \geq 4$. Let Σ be the set of places of bad reduction defined in 5.8 and let $\widetilde{V}$ be an irreducible $\mathbf{Q}$-linear representation of $\widetilde{G}$.*

Then, for each $v \notin \Sigma \cup \Sigma_p$, the Frobenius element $\tilde{\rho}_p(\mathrm{Fr}_v)$ acts semi-simply on $\widetilde{V}_p = \widetilde{V} \otimes \mathbf{Q}_p$ and its eigenvalues are algebraic integers with all complex absolute values equal to 1*.*

Proof. It suffices to prove the proposition for the faithful representation $\widetilde{V}$ constructed in 5.4 and 5.5 (according to the parity of k) and after replacing F by a finite extension. Let $\tilde{\rho}_p$ be lifted from the representation associated to the M-T decomposed and M-T unliftable abelian variety A/F, with Mumford–Tate group G_A. The derived group G_A^{der} is of the form $\mathrm{Res}_{K_0/\mathbf{Q}} G^{s,\mathrm{der}}$ and we have $\widetilde{G}^{\mathrm{der}} = \mathrm{Res}_{K_0/\mathbf{Q}} \widetilde{G}^{s,\mathrm{der}}$ where $\widetilde{G}^{s,\mathrm{der}}$ is the universal cover of $G^{s,\mathrm{der}}$. By construction, the representation $\widetilde{V}$ carries a structure of K_0-vector space compatible with the structure of a Weil restriction on $\widetilde{G}^{\mathrm{der}}$, so the action of $\widetilde{G}$ on $\widetilde{V}$ commutes with the action of K_0.

It is enough to prove the proposition for the representation of $\mathscr{G}_F$ on the tensor product $\widetilde{W}_p = \widetilde{V}_p \otimes_{K_{0,p}} \widetilde{V}_p$, where $K_{0,p} = K_0 \otimes_{\mathbf{Q}} \mathbf{Q}_p$. The representation of $\widetilde{G}^{\mathrm{der}}_{/\mathbf{Q}_p}$ on $\widetilde{W}_p$ factors through a representation of $G^{\mathrm{der}}_{A/\mathbf{Q}_p}$. In fact, $\widetilde{W}_p$ is a faithful representation of $G^{\mathrm{der}}_{A/\mathbf{Q}_p}$, but this plays no role in what follows. By theorem 4.9, $\widetilde{W}_p$ occurs in the p-adic realization of an abelian motive M. The proof of 4.9 even shows that we can find such a motive M in the subcategory of $(\mathbf{AV})_F$ generated by $h^1(A)$ and $(\mathbf{CM})_F$ so it follows that M has potentially good reduction at v for each $v \notin \Sigma$. After replacing F by a finite extension, we can assume that M has good reduction at all $v \notin \Sigma$. This implies that for any valuation $v \notin \Sigma \cup \Sigma_p$, the Frobenius element $\tilde{\rho}_p(\mathrm{Fr}_v)$ acts semi-simply on $\widetilde{W}_p$ and that each eigenvalue λ is an algebraic integer with all complex absolute values of the form $\mathbf{N}v^{w(\lambda)/2}$, where $\mathbf{N}v$ is the cardinal of the residue field of F at v and $w(\lambda)$ is an integer.

To show that all $w(\lambda)$ are equal to 0, it suffices to show that the Betti realization of M is pure of weight 0. This realization can be determined as follows. The cocharacter $\tilde{\mu}$ constructed in 5.6 defines a map $\tilde{h}_{\mathbf{C}} \colon S_{/\mathbf{C}} \to \widetilde{G}_{/\mathbf{C}}$ given by $\tilde{h}_{\mathbf{C}}(z, \bar{z}) = \tilde{\mu}(z)\overline{\tilde{\mu}(\bar{z})}$. This map descends to a map $\tilde{h} \colon S \to \widetilde{G}_{/\mathbf{R}}$ lifting $h_A^{\mathrm{ad}} \colon S \to G_A^{\mathrm{ad}}$ and these data define a Hodge structure on $\widetilde{V}$ on which K_0 acts by endomorphisms. The tensor product $\widetilde{W} = \widetilde{V} \otimes_{K_0} \widetilde{V}$ is a representation of G_A^{der}, and the Hodge structure on $\widetilde{W}$ derived from the Hodge structure on $\widetilde{V}$ is the Hodge structure on the Betti realization of the abelian motive M. Since $\tilde{\mu}$ and its complex conjugate are inverse to each other, it follows that the Hodge structure on $\widetilde{V}$ is of weight 0 and thus the same thing holds for $\widetilde{W}$. $\square$

6.9 We keep the notations of proposition 6.8. In particular, Σ is the set of places of bad reduction defined in 5.8. For the representation $\widetilde{V}$ however, we fix the $\mathbf{Q}$-linear representation of $\widetilde{G}$ constructed in 5.4 or 5.5 as in the proof of proposi-

tion 6.8. It follows from corollary 5.9 that, after replacing F by a finite extension, there is a system of weak geometric lifts $\tilde{\rho}_p\colon \mathscr{G}_F \to \widetilde{G}(\mathbf{Q}_p)$ of the $\rho_{A,p}$, for varying p. From now on we will assume that we dispose of such a system.

The rest of the paper concerns the variation of the characteristic polynomials of the $\tilde{\rho}_p(\mathrm{Fr}_v)$ on the $\widetilde{V}_p = \widetilde{V} \otimes \mathbf{Q}_p$, for a fixed valuation $v \notin \Sigma$ of F and varying p with $v(p) = 0$. We will deduce our result on the characteristic polynomials from the statement 6.12 which is independent of the choice of a representation of $\widetilde{G}$.

As in the proof of proposition 6.8, assume that $\tilde{\rho}_p$ is lifted from the representation associated to the Mumford–Tate decomposed and essentially M-T unliftable abelian variety A/F, with Mumford–Tate group G_A. Without loss of generality, we can assume that G_A is of the type constructed in remark 2.13.2. We also keep the notations $G_A^{\mathrm{der}} = \mathrm{Res}_{K_0/\mathbf{Q}} G^{s,\mathrm{der}}$ and $\widetilde{G}^{\mathrm{der}} = \mathrm{Res}_{K_0/\mathbf{Q}} \widetilde{G}^{s,\mathrm{der}}$, for a totally real number field K_0 and semi-simple groups $G^{s,\mathrm{der}}$ and $\widetilde{G}^{s,\mathrm{der}}$ over K_0. As explained in 2.13.2, there is a totally imaginary quadratic extension L of K_0 such that G_A is isogenous to a subgroup of $G_A^{\mathrm{der}} \times L^\times$. It follows from loc. cit. that L is contained in the centre of $\mathrm{End}^0(A_{/\mathbf{C}}) = \mathrm{End}(A_{/\mathbf{C}}) \otimes_{\mathbf{Z}} \mathbf{Q}$. In this case, there exists a linear algebraic group $\widetilde{G}^s$ over K_0 such that $\widetilde{G} = \mathrm{Res}_{K_0/\mathbf{Q}} \widetilde{G}^s$

Let $\widetilde{N} \subset \widetilde{G}^{\mathrm{der}} \subset \widetilde{G}$ be the centre of $\widetilde{G}^{\mathrm{der}}$. It is the kernel of the natural map $\widetilde{G} \to G' = \widetilde{G}^{\mathrm{ad}} \times \widetilde{G}^{\mathrm{ab}}$. There is an isomorphism $\widetilde{N} \cong \mathrm{Res}_{K_0/\mathbf{Q}} \widetilde{N}^s$, where $\widetilde{N}^s$ is the centre of $\widetilde{G}^{s,\mathrm{der}}$, a finite group scheme over K_0 of order 4, geometrically isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$ if k is even and to $\mathbf{Z}/4\mathbf{Z}$ if k is odd. Note that, by construction, each $\tilde{\rho}_p$ is determined up to a finite character $\mathscr{G}_F \to \widetilde{N}(\mathbf{Q}_p)$ which is unramified outside $\Sigma \cup \Sigma_p$. This ambiguity explains the fact that in the next proposition the characteristic polynomials may vary with p.

For each valuation v of F, each prime number p and each $\varepsilon \in \widetilde{N}(\mathbf{Q}_p)$, let $\widetilde{P}^v_{\varepsilon,p}(X) \in \mathbf{Q}_p[X]$ be the characteristic polynomial

$$\widetilde{P}^v_{\varepsilon,p}(X) = \det_{\widetilde{V}_p}\left(\varepsilon \tilde{\rho}_p(\mathrm{Fr}_v) - X \cdot \mathrm{id}\right).$$

We write $\widetilde{P}^v_p(X) = \widetilde{P}^v_{1,p}(X)$ for the characteristic polynomial of $\tilde{\rho}_p(\mathrm{Fr}_v)$.

6.10 Proposition. *Assume that for some, hence any, prime number ℓ, the rank of the Zariski closure of the image of $\rho_{A,\ell}$ is equal to the rank of G_A. Then there exist a set $\Sigma_{\mathrm{Fr}} \supset \Sigma$ of valuations of F of Dirichlet density* 0 *and, for each $v \notin \Sigma_{\mathrm{Fr}}$, a polynomial $\widetilde{P}^v \in \mathbf{Q}[X]$ such that for every prime number p with $v(p) = 0$, one has $\widetilde{P}^v_{\varepsilon,p} = \widetilde{P}^v$ for some $\varepsilon = \varepsilon_p \in \widetilde{N}(\mathbf{Q}_p)$.*

6.11 Corollary. *Let notations and hypotheses be as in the proposition. Let M^{nor} be a normal closure of $K_0/\mathbf{Q}$. For each prime number p, let M_p^{nor} be the image of M^{nor} in $\overline{\mathbf{Q}}_p$ and put $M_p = M_p^{\mathrm{nor}} \cap \mathbf{Q}_p$. Then, for every $v \notin \Sigma_{\mathrm{Fr}}$, the characteristic polynomial $\widetilde{P}_p^v$ of $\tilde{\rho}_p(\mathrm{Fr}_v)$ lies in $M_p[X]$.*

Proofs. The ℓ-independence of the rank of the Zariski closure of the image of $\rho_{A,\ell}$ follows from [Ser85, 2.2.4]. The corollary easily follows from the proposition. To prove the proposition we will reformulate the result in terms of the the quotient variety of $\widetilde{G}$ by a subgroup $\mathrm{Aut}' \subset \mathrm{Aut}(\widetilde{G}^{\mathrm{der}})$.

If $k \geq 5$, we put $\mathrm{Aut}'(\widetilde{G}^{s,\mathrm{der}}) = \mathrm{Aut}(\widetilde{G}^{s,\mathrm{der}})$ and $\mathrm{Out}'(\widetilde{G}^{s,\mathrm{der}}) = \mathrm{Out}(\widetilde{G}^{s,\mathrm{der}})$. If $k = 4$ then $\mathrm{Out}(\widetilde{G}^{s,\mathrm{der}}) \cong S_3$, acting naturally on the vertices $\alpha_1, \alpha_3, \alpha_4$ of the Dynkin diagram. Let $\mathrm{Out}' \subset \mathrm{Out}(\widetilde{G}^{s,\mathrm{der}})$ be the stabiliser of α_1 and $\mathrm{Aut}'(\widetilde{G}^{s,\mathrm{der}})$ the inverse image of Out' in $\mathrm{Aut}(\widetilde{G}^{s,\mathrm{der}})$.

In either case, $\mathrm{Aut}'(\widetilde{G}^{s,\mathrm{der}})$ is an extension of $\mathrm{Out}'(\widetilde{G}^{s,\mathrm{der}}) \cong \mathbf{Z}/2\mathbf{Z}$ by $\widetilde{G}^{s,\mathrm{ad}}$. It acts on the centre of $\widetilde{G}^{s,\mathrm{der}}$ through its quotient $\mathrm{Out}'(\widetilde{G}^{s,\mathrm{der}})$ and by going through the constructions in 5.2 to 5.5 (sub)case by (sub)case, it is not very difficult to check that this action extends to an action of $\mathrm{Out}'(\widetilde{G}^{s,\mathrm{der}})$ on the centre of $\widetilde{G}^s$. We thus obtain an action of $\mathrm{Aut}'(\widetilde{G}^{s,\mathrm{der}})$ on $\widetilde{G}^s$ and, taking Weil restrictions, an action of $\mathrm{Aut}' = \mathrm{Res}_{K_0/\mathbf{Q}}\mathrm{Aut}'(\widetilde{G}^{s,\mathrm{der}})$ on $\widetilde{G}$.

Let $\mathrm{Cl}(\widetilde{G})$ be the categorical quotient of $\widetilde{G}$ by this action, see [MF82, Chapter 1]. This means that if $R = \Gamma(\widetilde{G}, \mathscr{O}_{\widetilde{G}})$ is the affine coordinate ring of $\widetilde{G}$ then the quotient is given by $\mathrm{Cl}(\widetilde{G}) = \mathrm{Spec}(R^{\mathrm{Aut}'})$. For each prime number p, we denote by $\mathrm{Cl}(\tilde{\rho}_p)\colon \mathscr{G}_F \to \mathrm{Cl}(\widetilde{G})(\mathbf{Q}_p)$ the map deduced from $\tilde{\rho}_p$. The proposition 6.10 then follows from the following, more precise, statement. □

6.12 Proposition. *Assume that for some, hence any, prime number ℓ, the rank of the Zariski closure of the image of $\rho_{A,\ell}$ is equal to the rank of G_A. Then there are a set $\Sigma_{\mathrm{Fr}} \supset \Sigma$ of valuations of F of Dirichlet density 0 and elements $\mathrm{Cl}(\widetilde{\mathrm{Fr}}_v) \in \mathrm{Cl}(\widetilde{G})(\mathbf{Q})$, for $v \notin \Sigma_{\mathrm{Fr}}$, such that for every prime number p there exists $\varepsilon_p \in \widetilde{N}(\mathbf{Q}_p)$ such that the conjugacy class of $\varepsilon_p\tilde{\rho}_p(\mathrm{Fr}_v)$ is $\mathrm{Cl}(\widetilde{\mathrm{Fr}}_v)$.*

The proof requires several lemmas and the following notation. The group Aut' defined above also acts on G_A^{der}. As $\mathrm{Out}'(\widetilde{G}^{s,\mathrm{der}}) \subset \mathrm{Out}(G_A^{s,\mathrm{der}})$ acts trivially on the centre of $G_A^{s,\mathrm{der}}$, this action extends to an action of Aut' on G_A with trivial action on the centre. As above, we write $\mathrm{Cl}(G_A)$ for the categorical quotient of G_A by this action and let $\mathrm{Cl}(\rho_{A,p})\colon \mathscr{G}_F \to \mathrm{Cl}(G_A)(\mathbf{Q}_p)$ be the map induced by $\rho_{A,p}\colon \mathscr{G}_F \to G_A(\mathbf{Q}_p)$, for each prime number p.

Similarly, Aut' acts naturally on $\widetilde{G}^{\mathrm{ad}}$ and on $\widetilde{G}^{\mathrm{ab}}$ so we deduce an action on $G' = \widetilde{G}^{\mathrm{ad}} \times \widetilde{G}^{\mathrm{ab}}$. Let $\mathrm{Cl}(\widetilde{G}^{\mathrm{ad}})$ (resp. $\mathrm{Cl}(G')$) be the quotient of $\widetilde{G}^{\mathrm{ad}}$ (resp. G') by Aut'. The maps $G_A \to \widetilde{G}^{\mathrm{ad}} \to G'$ and $\widetilde{G} \to G'$ are Aut'-equivariant and therefore induce maps $\mathrm{Cl}(G_A) \to \mathrm{Cl}(G')$ and $\mathrm{Cl}(\widetilde{G}) \to \mathrm{Cl}(G')$.

Recall from 5.8 and corollary 5.9 that the $\tilde{\rho}_p$ lift the representations

$$\rho'_p = (\rho_p^{\mathrm{ad}}, \rho_p^{\mathrm{ab}}) \colon \mathscr{G}_F \to G'(\mathbf{Q}_p).$$

Let the $\mathrm{Cl}(\rho'_p) \colon \mathscr{G}_F \to \mathrm{Cl}(G')(\mathbf{Q}_p)$ be the induced maps.

6.13 Lemma. *For each $v \notin \Sigma$, there is an element*

$$\mathrm{Cl}(\mathrm{Fr}_{A,v}) \in \mathrm{Cl}(G_A)(\mathbf{Q})$$

such that for every prime number p with $v(p) = 0$ we have $\mathrm{Cl}(\rho_p)(\mathrm{Fr}_v) = \mathrm{Cl}(\mathrm{Fr}_{A,v})$.

Proof. We use the notation of 6.9. The hypotheses that G_A arises from the construction of 2.13.2 imply that $G_A^{\mathrm{der}} \times L^\times$ acts on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbf{C}), \mathbf{Q})$ and that this representation is a Weil restriction of $W \otimes_{K_0} V^s$, where W is the representation of $L^\times$ on L by left multiplication and V^s is a multiple of the representation of $G^{s,\mathrm{der}}$ of highest weight ϖ_1. It follows in particular that L lies in the centre of $\mathrm{End}^0(A_{/\mathbf{C}}) = \mathrm{End}(A_{/\mathbf{C}}) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Since $v \notin \Sigma$, the abelian variety A has good reduction A_v at v so we can identify $\mathrm{End}^0(A_{/\mathbf{C}})$ with a subalgebra of $\mathrm{End}^0(A_{\bar{v}})$. Here $A_{\bar{v}}$ is the base extension of A_v to the algebraic closure of the residue field of F at v. We obtain an embedding $L \subset \mathrm{End}^0(A_{\bar{v}})$. Let $\pi_v \colon A_v \to A_v$ be the Frobenius endomorphism. For each prime number p different from the residue characteristic at v, there is a canonical, hence L-equivariant, identification $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p) = \mathrm{H}_{\text{ét}}(A_{\bar{v}}, \mathbf{Q}_p)$. Under this identification, the action of $\rho_p(\mathrm{Fr}_v)$ on the left hand side corresponds to the action of π_v on the right hand side.

Since π_v is semi-simple and lies in the centre of $\mathrm{End}^0(A_v)$, the subalgebra $M = L[\pi_v] \subset \mathrm{End}^0(A_v)$ is a product of number fields M_i. Each M_i is of the form $\mathbf{Q}(\alpha_i)$ for some $\alpha_i \in \mathrm{End}^0(A_v)$ and it is a standard fact (cf. [Mum70, §19, theorem 4]) that the characteristic polynomial of α_i acting on $\mathrm{H}_{\text{ét}}(A_{\bar{v}}, \mathbf{Q}_p)$ has coefficients in $\mathbf{Q}$ and is independent of p. This implies that there exists an M-module U_v such that for every prime number p with $v(p) = 0$ there is an isomorphism $\mathrm{H}_{\text{ét}}(A_{\bar{v}}, \mathbf{Q}_p) \cong U_v \otimes_{\mathbf{Q}} \mathbf{Q}_p$ of $M \otimes_{\mathbf{Q}} \mathbf{Q}_p$-modules. Let $Q[X] \in L[X]$ be

the characteristic polynomial of $\pi_v \in M$ acting on U_v as an L-linear endomorphism. For every p with $v(p) = 0$, the characteristic polynomial of $\rho_{A,p}(\mathrm{Fr}_v)$, acting $L \otimes_{\mathbf{Q}} \mathbf{Q}_p$-linearly on $\mathrm{H}^1_{\text{ét}}(A_{\bar{F}}, \mathbf{Q}_p)$, is equal to Q. This result is due to Shimura, see [Shi67, 11.10.1].

Consider the map $G^{s,\mathrm{der}}_{/\overline{\mathbf{Q}}} \to \mathbf{A}^n_{/\overline{\mathbf{Q}}}$ corresponding to the characteristic polynomial in the representation with highest weight ϖ_1. It factors through the quotient of $G^{s,\mathrm{der}}$ for the action of $\mathrm{Aut}'(G^{s,\mathrm{der}})$, giving a map $\mathrm{Cl}(G_A) \to \mathrm{Res}_{L/\mathbf{Q}}\mathbf{A}^n_L$. This map is easily seen to be injective on geometric points. The fact that Q is the characteristic polynomial of $\rho_{A,p}(\mathrm{Fr}_v)$ for every p with $v(p) = 0$ implies that Q is the image in $\mathrm{Res}_{L/\mathbf{Q}}\mathbf{A}^n_L$ of an element $\mathrm{Cl}(\mathrm{Fr}_{A,v}) \in \mathrm{Cl}(G_A)(\mathbf{Q})$ and that this element $\mathrm{Cl}(\mathrm{Fr}_{A,v})$ verifies the condition of the lemma. □

6.14 Corollary. *For each $v \notin \Sigma$, there is an element*

$$\mathrm{Cl}(\mathrm{Fr}'_v) \in \mathrm{Cl}(G')(\mathbf{Q})$$

such that for every prime number p with $v(p) = 0$ we have $\mathrm{Cl}(\rho'_p)(\mathrm{Fr}_v) = \mathrm{Cl}(\mathrm{Fr}'_v)$.

Proof. By 5.8, the $\rho^{\mathrm{ab}}_p \colon \mathscr{G}_F \to \widetilde{G}^{\mathrm{ab}}(\mathbf{Q}_p)$ form a compatible system, so all $\rho^{\mathrm{ab}}_p(\mathrm{Fr}_v)$ are defined over $\mathbf{Q}$ and coincide. This gives rise to an element $\mathrm{Fr}^{\mathrm{ab}}_v \in \widetilde{G}^{\mathrm{ab}}(\mathbf{Q})$. On the adjoint side, let $\mathrm{Cl}(\widetilde{\mathrm{Fr}}^{\mathrm{ad}}_v) \in \mathrm{Cl}(\widetilde{G}^{\mathrm{ad}})(\mathbf{Q})$ be the image of $\mathrm{Cl}(\mathrm{Fr}_{A,v})$ and let $\mathrm{Cl}(\mathrm{Fr}'_v)$ be the image of $(\mathrm{Cl}(\widetilde{\mathrm{Fr}}^{\mathrm{ad}}_v), \mathrm{Fr}^{\mathrm{ab}}_v)$ in $\mathrm{Cl}(G')(\mathbf{Q})$.

For each p, the representation $\tilde{\rho}_p$ is a weak geometric lift of $\rho_{A,p}$ so the projection $\tilde{\rho}^{\mathrm{ad}}_p \colon \mathscr{G}_F \to \widetilde{G}^{\mathrm{ad}}(\mathbf{Q}_p)$ coincides with the composite of $\rho_{A,p}$ with the projection $G_A(\mathbf{Q}_p) \to \widetilde{G}^{\mathrm{ad}}(\mathbf{Q}_p)$. It follows that $\mathrm{Cl}(\mathrm{Fr}'_v)$ is the element promised by the corollary. □

6.15 Lemma. *Let M be either the group scheme $\widetilde{N} = \mathrm{Res}_{K_0/\mathbf{Q}}\widetilde{N}^s$ (the centre of $\widetilde{G}^{\mathrm{der}}$, cf. 6.9) or*

$$\mathrm{Out}'(\widetilde{G}) = \mathrm{Aut}'(\widetilde{G})/\widetilde{G}^{\mathrm{ad}} \cong \mathrm{Res}_{K_0/\mathbf{Q}}\mu_2$$

and let P be a co-finite set of prime numbers. Then the natural map

$$\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(\mathbf{Q}), M) \to \prod_{p \in P} \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(\mathbf{Q}_p), M)$$

is injective.

Proof. In the lemma, M denotes either $\widetilde{N}$ or $\mathrm{Out}'(\widetilde{G})$. We put $M' = \widetilde{N}^s$ in the former and $M' = \mu_2$ in the latter case. Let $f \colon \mathrm{Spec}(K_0) \to \mathrm{Spec}(\mathbf{Q})$ be the

natural morphism. This implies that $M = f_* M'$ as étale sheaves and since f_* is exact by [Mil80, II, Corollary 3.6], this gives an isomorphism

$$\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(\mathbf{Q}), M) \cong \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(K_0), M'),$$

cf. [Ser94, I. 2.5] for the interpretation in terms of Galois cohomology. Similarly, for every prime number p, there is an isomorphism

$$\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(\mathbf{Q}_p), M) \cong \prod_{\mathfrak{p}|p} \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(K_{0,\mathfrak{p}}), M'),$$

obtained from the above identification by base change to $\mathrm{Spec}(\mathbf{Q}_p)$. To prove the lemma, it is sufficient to prove the injectivity of the map

$$\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(K_0), M') \to \prod_{\mathfrak{p}\in P'} \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(K_{0,\mathfrak{p}}), M'),$$

where P' is the set of primes of K_0 lying over the rational primes in P. In the case where $M = \mathrm{Out}'(\widetilde{G})$, the group scheme $M' = \mu_2$ is constant and as $\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(K), \mu_2) \cong \mathrm{Hom}(\mathscr{G}_K, \mu_2(K))$ for any field K, the lemma follows.

In the case where $M' \cong \widetilde{N}^s$, there are two possibilities, M' is geometrically isomorphic to either $(\mathbf{Z}/2\mathbf{Z})^2$ or $\mathbf{Z}/4\mathbf{Z}$. In both cases the group scheme becomes trivial over an extension of degree (at most) 2. If M' is already trivial over K_0, then the above argument applies. If M' is geometrically isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$ and trivial over the quadratic extension $K \supset K_0$, then M' is a Weil restriction and the lemma follows by the same argument as before.

We are left with the case where $M'_K \cong (\mathbf{Z}/4\mathbf{Z})_K$ for a quadratic extension $K \supset K_0$. The proof in this remaining case, which also applies in the other cases where $M' \cong \widetilde{N}^s$, makes use of the long exact cohomology sequences associated to the short exact sequence $1 \to \mu_2 \to M' \to \mu_2 \to 1$ of étale sheaves on $\mathrm{Spec}(K_0)$. This gives a commutative diagram with exact rows,

$$\begin{tikzcd}
\mu_2(K_0) \arrow[r] \arrow[d] & \mathrm{H}^1(K_0, \mu_2) \arrow[r, "i"] \arrow[d, "\mathrm{loc}_{\mu_2}"] & \mathrm{H}^1(K_0, M') \arrow[r] \arrow[d, "\mathrm{loc}_{M'}"] & \mathrm{H}^1(K_0, \mu_2) \arrow[d, "\mathrm{loc}_{\mu_2}"] \\
\prod \mu_2(K_{0,\mathfrak{p}}) \arrow[r] & \prod \mathrm{H}^1(K_{0,\mathfrak{p}}, \mu_2) \arrow[r, "\prod i_{\mathfrak{p}}"] & \prod \mathrm{H}^1(K_{0,\mathfrak{p}}, M') \arrow[r] & \prod \mathrm{H}^1(K_{0,\mathfrak{p}}, \mu_2).
\end{tikzcd}$$

Here all H^1 are étale cohomology groups over the spectrum of the specified field. These groups can be identified with the corresponding Galois cohomology groups. The products in the second row are over all $\mathfrak{p} \in P'$.

The image of i identifies with $K_0^\times/\langle K_0^{\times 2}, \alpha\rangle$ for some $\alpha \in K_0^\times$, where $\langle K_0^{\times 2}, \alpha\rangle$ is the subgroup of $K_0^\times$ generated by α and the squares in $K_0^\times$. Similarly, the image of each $i_\mathfrak{p}$ is identified with $K_{0,\mathfrak{p}}^\times/\langle K_{0,\mathfrak{p}}^{\times 2}, \alpha\rangle$. If $x \in \ker \mathrm{loc}_{M'}$, then the injectivity of loc_{μ_2} implies that x lies in the image of i. Let $y \in K_0^\times$ represent the preimage of x in $K_0^\times/\langle K_0^{\times 2}, \alpha\rangle$. Since $x \in \ker \mathrm{loc}_{M'}$, the element y lies in $\langle K_{0,\mathfrak{p}}^{\times 2}, \alpha\rangle$ for each $\mathfrak{p} \in P'$, so y is a square in $K_0(\sqrt{\alpha})$ locally at each place above a $\mathfrak{p} \in P'$. It follows that y is a square in $K_0(\sqrt{\alpha})$, hence it maps to 1 in $K_0^\times/\langle K_0^{\times 2}, \alpha\rangle$. $\square$

Proof of proposition 6.12. The map $\widetilde{G} \to G'$ induces a map $\mathrm{pr}\colon \mathrm{Cl}(\widetilde{G}) \to \mathrm{Cl}(G')$. Obviously,

$$\mathrm{Cl}(\rho'_p) = \mathrm{pr}_{\mathbf{Q}_p} \circ \mathrm{Cl}(\tilde{\rho}_p)\colon \mathscr{G}_F \to \mathrm{Cl}(G')(\mathbf{Q}_p)$$

for each p.

For the groups $\widetilde{G}$ and G', let $\mathrm{Conj}(\widetilde{G})$ and $\mathrm{Conj}(G')$ be the varieties of geometric conjugacy classes, i. e. the quotients for the actions of $\widetilde{G}^{\mathrm{ad}}$. Note that $\mathrm{Cl}(\widetilde{G})$ (resp. $\mathrm{Cl}(G')$) is the quotient of $\mathrm{Conj}(\widetilde{G})$ (resp. $\mathrm{Conj}(G')$) by $\mathrm{Out}'(\widetilde{G})$ and that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Conj}(\widetilde{G}) & \longrightarrow & \mathrm{Cl}(\widetilde{G}) \\ \Big\downarrow & & \Big\downarrow \mathrm{pr} \\ \mathrm{Conj}(G') & \longrightarrow & \mathrm{Cl}(G'). \end{array}$$

There is a Zariski closed subset $B \subset \mathrm{Cl}(G')$ such that $\mathrm{Conj}(G') \to \mathrm{Cl}(G')$ and pr are both unramified outside B. Let the $\mathrm{Cl}(\mathrm{Fr}'_v)$ be as in corollary 6.14. The hypothesis on the rank of the Zariski closure of the image of $\rho_{A,\ell}$ and the Chebotatev density theorem imply that there is a set Σ_{Fr} of places of F of Dirichlet density 0 such that $\mathrm{Cl}(\mathrm{Fr}'_v) \in B(\mathbf{Q})$ if and only if $v \in \Sigma_{\mathrm{Fr}}$. To prove the proposition, it suffices to show that for every $v \notin \Sigma_{\mathrm{Fr}}$, the fibre of pr over $\mathrm{Cl}(\mathrm{Fr}'_v)$ contains an element $\mathrm{Cl}(\widetilde{\mathrm{Fr}}_v) \in \mathrm{Cl}(\widetilde{G})(\mathbf{Q})$.

Fix $v \notin \Sigma_{\mathrm{Fr}}$. The fibre of $\mathrm{Conj}(G') \to \mathrm{Cl}(G')$ over $\mathrm{Cl}(\mathrm{Fr}'_v)$ is a Out'-torsor which admits a $\mathbf{Q}_p$-valued point $\mathrm{Conj}(\rho'_p(\mathrm{Fr}_v))$ for almost every p. It follows from lemma 6.15 that there exists $\mathrm{Conj}(\mathrm{Fr}'_v) \in \mathrm{Conj}(G')(\mathbf{Q})$ above $\mathrm{Cl}(\mathrm{Fr}'_v)$. The fibre of $\mathrm{Conj}(\widetilde{G}) \to \mathrm{Conj}(G')$ over this element is a $\widetilde{N}$-torsor. As this fibre identifies with $\mathrm{pr}^{-1}(\mathrm{Cl}(\mathrm{Fr}'_v))$, this last fibre is also a $\widetilde{N}$-torsor. It has a $\mathbf{Q}_p$-valued point $\mathrm{Cl}(\tilde{\rho}_p(\mathrm{Fr}_v))$ for almost every p, so the promised existence of $\mathrm{Cl}(\widetilde{\mathrm{Fr}}_v) \in \mathrm{Cl}(\widetilde{G})(\mathbf{Q})$ follows from another application of lemma 6.15. $\square$

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